FIXED POINT THEOREMS FOR DUALISTIC EXPANDING MAPPING IN DUALISTIC PARTIAL METRIC SPACES

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Abstract

In this paper, we define dualistic expanding mappings in the setting of dualistic partial metric spaces analogous to expanding mappings in partial metric spaces. We establish some new fixed point theorems for dualistic expanding mappings defined on a dualistic partial metric space. Our result extends and generalizes some well-known results of [5] and [17]. We also provide an example which shows the usefulness of these dualistic expanding mappings.
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1 Introduction

Metric fixed point theory is playing an increasing role in mathematics because of its wide range of applications in applied mathematics and sciences. There have been a number of generalizations of the usual notion of a metric space. One such generalization is a partial metric space introduced and studied by Matthews [7]. He confirmed the precise relationship between partial metric spaces and the so-called weightable quasi-metric spaces. There are some generalizations of partial metrics. For example, O'Neill [14] proposed one significant change to Matthews' definition of the partial metric, and that was extending their range from $[0, \infty)$ to $(-\infty, \infty)$. According to [14], the partial metrics in the O'Neill sense will be called dualistic partial metric and a pair $(\mathfrak{D}, \eta^*)$ such that $\mathfrak{D}$ is a nonempty set and η^* is a dualistic partial metric on $\mathfrak{D}$ will be called a dualistic partial metric space. In this way, O'Neill developed several connections between partial metrics and the topological aspects of domain theory.

Contractive conditions have been started by studying Banach's contraction principle. These conditions have been used in various fixed point theorems for some generalized metric space. Then expansive conditions were introduced [17] and new fixed point results were obtained using expansive mappings. Some fixed point results have been still investigated using the notions of metric space and partial metric space for various contractive or expansive mappings. For more details, see [3], [4], [5], [6], [8] [15], [16], [19].

The aim of this paper is to prove some fixed point results under various dualistic expansive mappings in a dualistic partial metric space. Our result extends and generalizes some well-known results of [5] and [17]. Also, we verify our results with an example.

2 Preliminaries

We recall some mathematical basics and definitions to make this paper self-sufficient.

***Definition* 2.1** *(see* [7]*) Let $\mathfrak{D}$ be a non-empty set. A partial metric on $\mathfrak{D}$ is a function $\eta : \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ complying with following axioms, for all $\sigma, \varsigma, \nu \in \mathfrak{D}$*

$(\eta_1) \sigma = \varsigma \Leftrightarrow \eta(\sigma, \varsigma) = \eta(\sigma, \sigma) = \eta(\varsigma, \varsigma)$;

$(\eta_2)\ \eta(\sigma, \sigma) \le \eta(\sigma, \varsigma)$;

$(\eta_3)\ \eta(\sigma, \varsigma) = \eta(\varsigma, \sigma)$;

$(\eta_4)\ \eta(\sigma, \varsigma) \le \eta(\sigma, \nu) + \eta(\nu, \varsigma) - \eta(\nu, \nu)$

The pair $(\mathfrak{D}, \eta)$ is called a partial metric space.

***Definition* 2.2** *(see* [14]*) Let $\mathfrak{D}$ be a non-empty set. A dualistic partial metric on $\mathfrak{D}$ is a function $\eta^* : \mathfrak{D} \times \mathfrak{D} \to (-\infty, \infty)$ satisfying the following axioms, for all $\sigma, \varsigma, \nu \in \mathfrak{D}$*

$(\eta_1^*)\ \sigma = \varsigma \Leftrightarrow \eta^*(\sigma, \varsigma) = \eta^*(\sigma, \sigma) = \eta^*(\varsigma, \varsigma)$;

(η_2^*) $\eta^*(\sigma,\sigma) \le \eta^*(\sigma,\varsigma)$;
(η_3^*) $\eta^*(\sigma,\varsigma) = \eta^*(\varsigma,\sigma)$;
(η_4^*) $\eta^*(\sigma,\nu) + \eta^*(\varsigma,\varsigma) \le \eta^*(\sigma,\varsigma) + \eta^*(\varsigma,\nu)$
The pair $(\mathfrak{D},\eta^*)$ is called a dualistic partial metric space.

***Remark* 2.1** *Noting that each partial metric is a dualistic partial metric, but the converse is false. Indeed, define η^* on $(-\infty,\infty)$ as $\eta^*(\sigma,\varsigma) = \max\{\sigma,\varsigma\}, \forall\sigma,\varsigma \in (-\infty,\infty)$. Obviously, η^* is a dualistic partial metric on $(-\infty,\infty)$. Since $\eta^*(\sigma,\varsigma) < 0 \in [0,\infty), \forall\sigma,\varsigma \in (-\infty,0)$ and then η^* is not a partial metric on $(-\infty,\infty)$. This confirms our remark.*

***Example* 2.1** *(see* [10], [14]*)*

1. *Define $d_{\eta^*} : \mathfrak{D}\times\mathfrak{D} \to [0,\infty)$ by $d_{\eta^*}(\sigma,\varsigma) = d(\sigma,\varsigma)+b$, where d is a metric on a nonempty set $\mathfrak{D}$ and $b \in (-\infty,\infty)$ is arbitrary constant, then it is easy to check that η_d^* verifies axioms $(\eta_1^*) - (\eta_4^*)$ and hence $(\mathfrak{D},\eta^*)$ is a dualistic partial metric space.*
2. *Let η be a partial metric defined on a non empty set $\mathfrak{D}$. The function $\eta^* : \mathfrak{D}\times\mathfrak{D} \to (-\infty,\infty)$ defined by $\eta^*(\sigma,\varsigma) = \eta(\sigma,\varsigma) - \eta(\sigma,\sigma) - \eta(\varsigma,\varsigma)$ satisfies the axioms $(\eta_1^*) - (\eta_4^*)$ and so it defines a dualistic partial metric on $\mathfrak{D}$. Note that $\eta^*(\sigma,\varsigma)$ may have negative values.*
3. *Let $\mathfrak{D} = (-\infty,\infty)$. Define $\eta^* : \mathfrak{D}\times\mathfrak{D} \to (-\infty,\infty)$ by $\eta^*(\sigma,\varsigma) = |\sigma-\varsigma|$ if $\sigma \ne \varsigma$ and $\eta^*(\sigma,\varsigma) = -\beta$ if $\sigma = \varsigma$ and $\beta > 0$. We can easily see that η^* is a dualistic partial metric on $\mathfrak{D}$.*

O'Neill [14] *established that each dualistic partial metric η^* on $\mathfrak{D}$ generates a $\mathcal{T}_0$ topology $\tau(\eta^*)$ on $\mathfrak{D}$ having a base, the family of η^* -balls $\{B_{\eta^*}(\sigma,\varepsilon) \in \sigma \in \mathfrak{D}, \varepsilon > 0\}$, where*

(2.1) $B_{\eta^*}(\sigma,\varepsilon) = \{\varsigma \in \mathfrak{D} \in \eta^*(\sigma,\varsigma) < \eta^*(\sigma,\sigma) + \varepsilon\}$.

If $(\mathfrak{D},\eta^*)$ is a dualistic partial metric space, then the function $d_{\eta^*} : \mathfrak{D}\times\mathfrak{D} \to [0,\infty)$ defined by

(2.2) $d_{\eta^*}(\sigma,\varsigma) = \eta^*(\sigma,\varsigma) - \eta^*(\sigma,\sigma)$

defines a quasi-metric on A such that $\tau(\eta^*) = \tau(d_{\eta^*})$ and

(2.3) $d^s_{\eta^*}(\sigma,\varsigma) = \max\{d_{\eta^*}(\sigma,\varsigma), d_{\eta^*}(\varsigma,\sigma)\}$

defines a metric on $\mathfrak{D}$.

***Definition* 2.3** *(see* [13]*) Let $(\mathfrak{D},\eta^*)$ be a dualistic partial metric space.*

1. *A sequence $\{\sigma_n\}$ in $\mathfrak{D}$ is said to converge or to be convergent if there is a $\sigma \in \mathfrak{D}$ such that $\lim_{n\to\infty}\eta^*(\sigma_n,\sigma) = \eta^*(\sigma,\sigma)$. σ is called the limit of $\{\sigma_n\}$ and we write $\sigma_n \to \sigma$.*
2. *A sequence $\{\sigma_n\}$ in $\mathfrak{D}$ is said to be Cauchy sequence if $\lim_{n,m\to\infty}\eta^*(\sigma_n,\sigma_m)$ exists and is finite.*
3. *A dualistic partial metric space $\mathfrak{D} = (\mathfrak{D},\eta^*)$ is said to be complete if every Cauchy sequence $\{\sigma_n\}$ in $\mathfrak{D}$ converges, with respect to $\tau(\eta^*)$, to a point $\sigma \in \mathfrak{D}$ such that $\eta^*(\sigma,\sigma) = \lim_{n,m\to\infty}\eta^*(\sigma_n,\sigma_m)$.*

Remark* 2.2** *For a sequence, convergence with respect to metric space may not imply convergence with respect to dualistic partial metric space. Indeed, if we take $\beta = 1$ and $\{\sigma_n = \frac{1-n}{n} : n \ge 1\}_{n\in N} \in \mathfrak{D}$ as in **Example 2.1 (3)*. Mention that $\lim_{n\to\infty} d(\sigma_n,-1) = -1$ and therefore, $\sigma_n \to -1$ with respect to d . On the other hand, we make a conclusion that $\sigma_n \in -1$ with respect to η^* because $\lim_{n\to\infty}\eta^*(\sigma_n,-1) = \lim_{n\to\infty}\eta^*|\sigma_n-(-1)| = \lim_{n\to\infty}|\frac{1-n}{n}+1| = 0$ and $\eta^*(-1,-1) = -1$.*

***Lemma* 2.1** *(see* [13]*) Let $(\mathfrak{D},\eta^*)$ be a dualistic partial metric space.*

1. *Every Cauchy sequence in $(\mathfrak{D},d^s_{\eta^*})$ is also a Cauchy sequence in $(\mathfrak{D},\eta^*)$.*
2. *A dualistic partial metric $(\mathfrak{D},\eta^*)$ is complete if and only if the induced metric space $(\mathfrak{D},d^s_{\eta^*})$ is complete.*
3. *A sequence $\{\sigma_n\}$ in $\mathfrak{D}$ converges to a point $\sigma \in \mathfrak{D}$ with respect to $\tau(d^s_{\eta^*})$ if and only if $\eta^*(\sigma,\sigma) = \lim_{n\to\infty}\eta^*(\sigma_n,\sigma) = \lim_{n\to\infty}\eta^*(\sigma_n,\sigma_m)$.*

3 Main Results

First we define the concept of expanding mapping in dualistic partial metric space.

***Definition* 3.1** *Let $(\mathfrak{D},\eta^*)$ be a dualistic partial metric space and $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$. Then $\mathcal{T}$ is called a dualistic expanding mapping, if for every $\sigma,\varsigma \in \mathfrak{D}$, there exists a number $\lambda > 1$ such that*

(3.1) $|\eta^*(\mathcal{T}\sigma,\mathcal{T}\varsigma)| \ge \lambda|\eta^*(\sigma,\varsigma)|$

Now, we investigate a unique fixed point of Huang et al. [5] *type dualistic expanding mappings.*

***Theorem* 3.1** *Let $(\mathfrak{D}, \eta^*)$ be a complete dualistic partial metric space and $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$ be a surjection. Suppose that there exist real numbers a, b, c satisfying $b, c \geq 0$ and $a > 1$ such that*

(3.2) $|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)| \geq a|\eta^*(\sigma, \varsigma)| + b|\eta^*(\sigma, \mathcal{T}\sigma)| + c|\eta^*(\varsigma, \mathcal{T}\varsigma)| \ \forall \sigma, \varsigma \in \mathfrak{D}$,

then $\mathcal{T}$ has a unique fixed point.

Proof. Since $\mathcal{T}$ be a surjective self-mapping on $\mathfrak{D}$, let us denote the inverse mapping of $\mathcal{T}$ by $\mathcal{F}$. Let σ_0 be an initial point in $\mathfrak{D}$ and define the sequence $\{\sigma_n\}_{n\in N}$ as follows:

(3.3) $\sigma_1 = \mathcal{F}\sigma_0, \sigma_2 = \mathcal{F}\sigma_1 = \mathcal{F}^2\sigma_0, \ldots, \sigma_{n+1} = \mathcal{F}\sigma_n = \mathcal{F}^{n+1}\sigma_0, \ldots.$

Without loss of generality, we assume that $\sigma_n \neq \sigma_{n+1}, \forall n \in \mathbb{N}$ (otherwise, if there exists some n_0 such that $\sigma_{n_0} = \sigma_{n_0+1} = \mathcal{T}\sigma_{n_0}$, then σ_{n_0} is a fixed of $\mathcal{T}$, so the proof is completed. It follows that from condition (3.2),

$$\begin{aligned}|\eta^*(\sigma_{n-1}, \sigma_n)| &= |\eta^*(\mathcal{T}\mathcal{T}^{-1}\sigma_{n-1}, \mathcal{T}\mathcal{T}^{-1}\sigma_n)| \\ &\geq a|\eta^*(\mathcal{T}^{-1}\sigma_{n-1}, \mathcal{T}^{-1}\sigma_n)| + b|\eta^*(\mathcal{T}^{-1}\sigma_{n-1}, \mathcal{T}\mathcal{T}^{-1}\sigma_{n-1})| + c|\eta^*(\mathcal{T}^{-1}\sigma_n, \mathcal{T}\mathcal{T}^{-1}\sigma_n)| \\ &= a|\eta^*(\mathcal{F}\sigma_{n-1}, \mathcal{F}\sigma_n)| + b|\eta^*(\mathcal{F}\sigma_{n-1}, \sigma_{n-1})| + c|\eta^*(\mathcal{F}\sigma_n, \sigma_n)| \\ &= a|\eta^*(\sigma_n, \sigma_{n+1})| + b|\eta^*(\sigma_n, \sigma_{n-1})| + c|\eta^*(\sigma_{n+1}, \sigma_n)|,\end{aligned}$$

which implies that

(3.4) $(1 - b)|\eta^*(\sigma_{n-1}\sigma_n)| \geq (a + c)|\eta^*(\sigma_n, \sigma_{n+1})|.$

Clearly, we have $a + c \neq 0$. Hence, we obtain

(3.5) $|\eta^*(\sigma_n, \sigma_{n+1})| \leq \dfrac{1-b}{a+c}|\eta^*(\sigma_{n-1}\sigma_n)|.$

If we put $\gamma = \frac{1-b}{a+c}$, then we get $\gamma < 1$, since $a + b + c > 1$ and repeating arguments given above, we have

(3.6) $|\eta^*(\sigma_n, \sigma_{n+1})| \leq \gamma^n|\eta^*(\sigma_0, \sigma_1)|.$

Now,

$$\begin{aligned}|\eta^*(\sigma_n, \sigma_n)| &= |\eta^*(\mathcal{T}\mathcal{T}^{-1}\sigma_n, \mathcal{T}\mathcal{T}^{-1}\sigma_n)| \\ &\geq a|\eta^*(\mathcal{T}^{-1}\sigma_n, \mathcal{T}^{-1}\sigma_n)| + b|\eta^*(\mathcal{T}^{-1}\sigma_n, \mathcal{T}\mathcal{T}^{-1}\sigma_n)| + c|\eta^*(\mathcal{T}^{-1}\sigma_n, \mathcal{T}\mathcal{T}^{-1}\sigma_n)| \\ &= a|\eta^*(\mathcal{F}\sigma_n, \mathcal{F}\sigma_n)| + b|\eta^*(\mathcal{F}\sigma_n, \sigma_n)| + c|\eta^*(\mathcal{F}\sigma_n, \sigma_n)| \\ &= a|\eta^*(\sigma_{n+1}, \sigma_{n+1})| + b|\eta^*(\sigma_{n+1}, \sigma_n)| + c|\eta^*(\sigma_{n+1}, \sigma_n)|.\end{aligned}$$

The last inequality gives

(3.7) $|\eta^*(\sigma_{n+1}, \sigma_{n+1})| \leq \dfrac{1}{a}|\eta^*(\sigma_n, \sigma_n)| - \dfrac{(b+c)}{a}|\eta^*(\sigma_{n+1}, \sigma_n)|.$

Due to inequality (3.6), we have

(3.8) $|\eta^*(\sigma_{n+1}, \sigma_{n+1})| \leq \dfrac{1}{a}|\eta^*(\sigma_n, \sigma_n)| - \dfrac{(b+c)}{a}\gamma^n|\eta^*(\sigma_0, \sigma_1)|.$

Similarly,

(3.9) $|\eta^*(\sigma_n, \sigma_n)| \leq \dfrac{1}{a}|\eta^*(\sigma_{n-1}, \sigma_{n-1})| - \dfrac{(b+c)}{a}\gamma^{n-1}|\eta^*(\sigma_0, \sigma_1)|.$

The inequality (3.6) implies that

$$\begin{aligned}|\eta^*(\sigma_{n+1}, \sigma_{n+1})| &\leq \frac{1}{a}\{\frac{1}{a}|\eta^*(\sigma_{n-1}, \sigma_{n-1})| - \frac{(b+c)}{a}\gamma^{n-1}|\eta^*(\sigma_0, \sigma_1)|\} - \tfrac{(b+c)}{a}\gamma^n|\eta^*(\sigma_0, \sigma_1)| \\ &= \frac{1}{a^2}|\eta^*(\sigma_{n-1}, \sigma_{n-1})| - \frac{(b+c)}{a^2}\gamma^{n-1}|\eta^*(\sigma_0, \sigma_1)| - \frac{(b+c)}{a}\gamma^n|\eta^*(\sigma_0, \sigma_1)| \\ &= \frac{1}{a^2}|\eta^*(\sigma_{n-1}, \sigma_{n-1})| - (b+c)[\frac{\gamma^{n-1}}{a^2} + \frac{\gamma^n}{a}]|\eta^*(\sigma_0, \sigma_1)| \\ &\leq \frac{1}{a^3}|\eta^*(\sigma_{n-2}, \sigma_{n-2})| - (b+c)[\frac{\gamma^{n-2}}{a^3} + \frac{\gamma^{n-1}}{a^2} + \frac{\gamma^n}{a}]|\eta^*(\sigma_0, \sigma_1)|.\end{aligned}$$

Continuing further, we get

(3.10) $|\eta^*(\sigma_{n+1}, \sigma_{n+1})| \leq \dfrac{1}{a^{n+1}}|\eta^*(\sigma_0, \sigma_0)| - (b+c)[\dfrac{1}{a^{n+1}} + \dfrac{\gamma}{a^n} + \ldots + \dfrac{\gamma^n}{a}]|\eta^*(\sigma_0, \sigma_1)|$

$= \delta^{n+1}|\eta^*(\sigma_0,\sigma_0)| - (b+c)\rho^{n+1}|\eta^*(\sigma_0,\sigma_1)|$

$\leq \delta^{n+1}|\eta^*(\sigma_0,\sigma_0)| + \rho^{n+1}|\eta^*(\sigma_0,\sigma_1)|,$

where $\delta = \frac{1}{a}$ and $\rho^{n+1} = \delta^{n+1} + \gamma\delta^n + \ldots + \gamma^{n-1}\delta^2 + \gamma^n\delta$.

We deduce from (2.2) that

$$(3.11)\quad d_{\eta^*}(\sigma_n,\sigma_{n+1}) \leq |\eta^*(\sigma_n,\sigma_{n+1})| - \eta^*(\sigma_n,\sigma_n)$$
$$\leq |\eta^*(\sigma_n,\sigma_{n+1})| + |\eta^*(\sigma_n,\sigma_n)|$$
$$\leq \gamma^n|\eta^*(\sigma_0,\sigma_1)| + \delta^n|\eta^*(\sigma_0,\sigma_0)| + \rho^n|\eta^*(\sigma_0,\sigma_1)|$$
$$= (\gamma^n + \rho^n)|\eta^*(\sigma_0,\sigma_1)| + \delta^n|\eta^*(\sigma_0,\sigma_0)|.$$

Now, for $m > n$, we have

$$d_{\eta^*}(\sigma_n,\sigma_m) \leq d_{\eta^*}(\sigma_n,\sigma_{n+1}) + d_{\eta^*}(\sigma_{n+1},\sigma_{n+2}) + \ldots + d_{\eta^*}(\sigma_{m-1},\sigma_m)$$
$$\leq (\gamma^n + \rho^n)|\eta^*(\sigma_0,\sigma_1)| + \delta^n|\eta^*(\sigma_0,\sigma_0)| + (\gamma^{n+1} + \rho^{n+1})|\eta^*(\sigma_0,\sigma_1)|$$
$$+\delta^{n+1}|\eta^*(\sigma_0,\sigma_0)| + \ldots + (\gamma^{m-1} + \rho^{m-1})|\eta^*(\sigma_0,\sigma_1)| + \delta^{m-1}|\eta^*(\sigma_0,\sigma_0)|$$
$$= (\gamma^n + \gamma^{n+1} + \ldots + \gamma^{m-1})|\eta^*(\sigma_0,\sigma_1)|$$
$$+(\rho^n + \rho^{n+1} + \ldots + \rho^{m-1})|\eta^*(\sigma_0,\sigma_1)|$$
$$+(\delta^n + \delta^{n+1} + \ldots + \delta^{m-1})|\eta^*(\sigma_0,\sigma_0)|$$
$$\leq (\gamma^n + \gamma^{n+1} + \ldots + \gamma^{m-1} + \ldots)|\eta^*(\sigma_0,\sigma_1)|$$
$$+(\rho^n + \rho^{n+1} + \ldots + \rho^{m-1} + \ldots)|\eta^*(\sigma_0,\sigma_1)|$$
$$+(\delta^n + \delta^{n+1} + \ldots + \delta^{m-1} + \ldots)|\eta^*(\sigma_0,\sigma_0)|$$
$$= \frac{\gamma^n}{1-\gamma}|\eta^*(\sigma_0,\sigma_1)| + \frac{\rho^n}{1-\rho}|\eta^*(\sigma_0,\sigma_1)| + \frac{\delta^n}{1-\delta}|\eta^*(\sigma_0,\sigma_0)|.$$

Hence

$$(3.12)\quad d_{\eta^*}(\sigma_n,\sigma_m) \leq \frac{\gamma^n}{1-\gamma}|\eta^*(\sigma_0,\sigma_1)| + \frac{\rho^n}{1-\rho}|\eta^*(\sigma_0,\sigma_1)| + \frac{\delta^n}{1-\delta}|\eta^*(\sigma_0,\sigma_0)|.$$

As $m,n \to \infty, d^s_{\eta^*}(\sigma_n,\sigma_m) = \max\{d_{\eta^*}(\sigma_n,\sigma_m), d_{\eta^*}(\sigma_m,\sigma_n)\} \to 0$, thus, $\{\sigma_n\}$ is a Cauchy sequence in $(\mathfrak{D}, d^s_{\eta^*})$. Since $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space, by ***Lemma* 2.1** (2), $(\mathfrak{D}, d^s_{\eta^*})$ is a complete metric space. Thus, there exists $\nu \in (\mathfrak{D}, d^s_{\eta^*})$ such that $\sigma_n \to \nu$ as $n \to \infty$, that is $\lim_{n\to\infty} d_{\eta^*}(\sigma_n,\nu) = 0$ and by ***Lemma* 2.1** (3), we know that

$$(3.13)\quad \eta^*(\nu,\nu) = \lim_{n\to\infty}\eta^*(\sigma_n,\nu) = \lim_{n\to\infty}\eta^*(\sigma_n,\sigma_m).$$

Since, $\lim_{n\to\infty} d_{\eta^*}(\sigma_n,\nu) = 0$, by (2.2) and (3.27), we have

$$(3.14)\quad \eta^*(\nu,\nu) = \lim_{n\to\infty}\eta^*(\sigma_n,\nu) = \lim_{n\to\infty}\eta^*(\sigma_n,\sigma_m) = 0.$$

This shows that $\{\sigma_n\}$ is a Cauchy sequence converging to $\nu \in (\mathfrak{D},\eta^*)$. Using the surjectivity hypothesis, there exists a point $\omega \in \mathfrak{D}$ such that $\nu = \mathcal{T}\omega$. We shall show that ν is a fixed point of $\mathcal{T}$. From condition (3.2), we have

$$(3.15)\quad |\eta^*(\sigma_n,\nu)| = |\eta^*(\mathcal{T}\sigma_{n+1},\mathcal{T}\omega)|$$
$$\geq a|\eta^*(\sigma_{n+1},\omega)| + b|\eta^*(\sigma_{n+1},\mathcal{T}\sigma_{n+1})| + c|\eta^*(\omega,\mathcal{T}\omega)|$$
$$= a|\eta^*(\sigma_{n+1},\omega)| + b|\eta^*(\sigma_{n+1},\sigma_n)| + c|\eta^*(\omega,\nu)|.$$

Applying limit as $n \to \infty$ and using equation (3.14), we have

$$0 \geq (a+c)|\eta^*(\nu,\omega)|,$$

which implies that $|\eta^*(\nu,\omega)| = 0$ and then $\eta^*(\nu,\omega) = 0$.

Again from (3.2), we have

$$|\eta^*(\nu,\nu)| = |\eta^*(\mathcal{T}\omega,\mathcal{T}\omega)|$$
$$\geq a|\eta^*(\omega,\omega)| + b|\eta^*(\omega,\mathcal{T}\omega)| + c|\eta^*(\omega,\mathcal{T}\omega)|$$
$$= a|\eta^*(\omega,\omega)| + b|\eta^*(\omega,\nu)| + c|\eta^*(\omega,\nu)|.$$

Since $a > 0, \eta^*(\nu,\omega) = 0 = \eta^*(\nu,\nu)$, we get $|\eta^*(\omega,\omega)| = 0$ and then $\eta^*(\omega,\omega) = 0$. Thus

(3.16) $\eta^*(\omega,\omega) = \eta^*(\nu,\nu) = \eta^*(\nu,\omega)$.

By using axiom (η_1^*) , we have $\omega = \nu$. This shows that ν is a fixed point of $\mathcal{T}$. To prove the uniqueness of ν , suppose that ν^* is another fixed point of $\mathcal{T}$, then $\mathcal{T}\nu^* = \nu^*$ and $\eta^*(\nu^*,\nu^*) = 0$. By (3.2), we obtain

(3.17)
$$\begin{aligned}|\eta^*(\nu,\nu^*)| &= |\eta^*(\mathcal{T}\nu,\mathcal{T}\nu^*)| \\ &\geq a|\eta^*(\nu,\nu^*)| + b|\eta^*(\nu,\mathcal{T}\nu)| + c|\eta^*(\nu^*,\mathcal{T}\nu^*)| \\ &= a|\eta^*(\nu,\nu^*)| + b|\eta^*(\nu,\nu)| + c|\eta^*(\nu^*,\nu^*)|,\end{aligned}$$

which implies that $\eta^*(\nu,\nu^*) = 0$, since $a > 0$.

$$\eta^*(\nu,\nu^*) = \eta^*(\nu,\nu) = \eta^*(\nu^*,\nu^*).$$

By (η_1^*) , we have $\nu = \nu^*$. Consequently, $\mathcal{T}$ has unique fixed point ν .

The following ***Corollary* 3.1** corresponds to the unique fixed point result of Wang *et al.* [17] type dualistic expansion.

***Corollary* 3.1** *Let $(\mathfrak{D},\eta^*)$ be a complete dualistic partial metric space and let $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$ be a dualistic expanding surjection. Then $\mathcal{T}$ has a unique fixed point.*

Proof. If in ***Theorem* 3.1**, we put $a = \lambda, b = c = 0$, then $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$ be a dualistic expanding surjection, so arguments follow the same lines as in the proof of ***Theorem* 3.1**.

***Corollary* 3.2** *Let $(\mathfrak{D},\eta^*)$ be a complete dualistic partial metric space and $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$ be a surjection. Suppose that there exist a positive integer n and a constant $\lambda > 1$ such that*

(3.18) $|\eta^*(\mathcal{T}^n\sigma,\mathcal{T}^n\varsigma)| \geq \lambda|\eta^*(\sigma,\varsigma)|, \ \forall \sigma,\varsigma \in \mathfrak{D}$,

then $\mathcal{T}$ has a unique fixed point.

Proof. From ***Theorem* 3.1**, $\mathcal{T}^n$ has a unique fixed point ν . But $\mathcal{T}^n\mathcal{T}\nu = \mathcal{T}\mathcal{T}^n\nu = \mathcal{T}\nu$, so $\mathcal{T}\nu$ is also a fixed point of $\mathcal{T}^n$. By uniqueness of limit, $\mathcal{T}\nu = \nu$. Hence ν is a fixed point of $\mathcal{T}$. Since the fixed point of $\mathcal{T}$ is also fixed point of $\mathcal{T}^n$, the fixed point of $\mathcal{T}$ is unique.

***Corollary* 3.3** *(Corollary* 2.1 *of Huang et al.* [5]*) Let $(\mathfrak{D},\eta)$ be a complete partial metric space and $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$ be a surjection. Suppose that there exists $\lambda > 1$ such that*

(3.19) $\eta(\mathcal{T}\sigma,\mathcal{T}\varsigma) \geq \lambda\eta(\sigma,\varsigma) \ \forall \sigma,\varsigma \in \mathfrak{D}$,

then $\mathcal{T}$ has a unique fixed point.

Proof. Since the restriction of a dualistic partial metric $\eta^*|_{[0,\infty)} = \eta$ is a partial metric, so arguments follow the same lines as in the proof of ***Theorem* 3.1**.

***Corollary* 3.4** *(Theorem* 2.1 *of Huang et al .* [5]*) Let $(\mathfrak{D},\eta)$ be a complete partial metric space and $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$ be a surjection. Suppose that there exist real numbers a,b,c satisfying $b,c \geq 0$ and $a > 1$ such that*

(3.20) $\eta(\mathcal{T}\sigma,\mathcal{T}\varsigma) \geq a\eta(\sigma,\varsigma) + b\eta(\sigma,\mathcal{T}\sigma) + c\eta(\varsigma,\mathcal{T}\varsigma), \ \forall \sigma,\varsigma \in \mathfrak{D}$,

then $\mathcal{T}$ has a unique fixed point.

Proof. Set $\eta^*|_{[0,\infty)} = \eta$ and arguments follow the same lines as in the proof of ***Theorem* 3.1.**

***Observations* 3.1**

1. Usually the range of a dualistic partial metric η^* is $(-\infty,\infty)$ but if we replace $(-\infty,\infty)$ by $[0,\infty)$, then η^* is identical to a partial metric η and hence ***Theorem* 3.1** is applicable in the setting of partial metric space.
2. If we set $\eta(\sigma,\sigma) = 0$ in ***Corollary* 3.3** and ***Corollary* 3.4**, we retrieve corresponding theorems in metric spaces.
3. Our main result extends and generalizes some well-known results of Huang *et al.* [5] and Wang *et al.* [17].

4 Example

In this section, we give an example in support of our main result.

***Example* 4.1** *Define* $\eta^* : (-\infty, 0]) \times (-\infty, 0]) \to (-\infty, \infty)$ *by* $\eta^*(\sigma, \varsigma) = \max\{\sigma, \varsigma\}$. *It is easy to check that* $(-\infty, 0], \eta^*)$ *is a complete dualistic partial metric space. Define* $\mathcal{T} : (-\infty, 0]) \to (-\infty, 0])$ *as* $\mathcal{T}\sigma = 6\sigma, \forall \sigma \in (-\infty, 0])$. *Further, for all* $\sigma, \varsigma \in (-\infty, 0])$ *with* $\sigma \geq \varsigma$, *and* $a = 2, b = 2, c = 1$, *we have*

$$\begin{aligned} |\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)| &= |\max\{6\sigma, 6\varsigma\}| = |6\sigma| \\ &> 2|\sigma| + 2|\frac{\sigma}{2}| + |\frac{\varsigma}{2}| \\ &= 2|\max\{\sigma, \varsigma\}| + 2|\max\{\sigma, \frac{\sigma}{2}\}| + |\max\{\varsigma, \frac{\varsigma}{2}\}| \\ &= 2|\eta^*(\sigma, \varsigma)| + 2|\eta^*(\sigma, \mathcal{T}\sigma)| + |\eta^*(\varsigma, \mathcal{T}\varsigma)|. \end{aligned}$$

Clearly, (3.2) *is satisfied and* $\mathcal{T}$ *is a self-surjection on* $((-\infty, 0])$. *In the view of* **Theorem 3.1**, $\mathcal{T}$ *has a unique fixed point in* $((-\infty, 0])$, *indeed* $T0 = 0$. *Also*

$$|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)| = |\max\{6\sigma, 6\varsigma\}| = |6\sigma| > \lambda|\sigma| = \lambda|\max\{\sigma, \varsigma\}| = \lambda|\eta^*(\sigma, \varsigma)|$$

for $1 < \lambda < 6$ *and* $\forall \sigma, \varsigma \in (-\infty, 0])$ *with* $\sigma \geq \varsigma$. *Thus,* $\mathcal{T}$ *is a dualistic expanding surjective self-mapping on* $((-\infty, 0])$. *From,* **Corollary 3.1**, $T0 = 0 \in (-\infty, 0])$ *is unique.*

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