

COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS IN A -METRIC SPACES

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ABSTRACT

In this article, we established some fixed point results for expansive mappings on A -metric spaces. Finally, the example is presented to support the new theorem proved. Our results extend/generalize many pre-existing results in literature.

KEYWORDS: A -metric space; expansive mapping; fixed point.

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1. Introduction

Fixed point theory has great importance in science and mathematics. Since this area has been developed very fast over the past two decades due to huge applications in various fields such as nonlinear analysis, topology and engineering problems, it has attracted considerable attention from researchers. The study of expansive mappings is a very interesting research area in the fixed point theory. Wang et al. [36] proved some fixed point theorems for expansion mappings, which correspond to some contractive mappings in metric spaces. In 1992, Daffer and Kaneko [8] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for

two mappings in complete metric spaces. In 1989, Bakhtin [4] introduced the concept of a b-metric space as a generalization of metric spaces, in which many researchers treated the fixed point theory. In 1993, Czerwik [6, 7] extended many results related to the b-metric spaces. In 1994, Matthews [24] introduced the concept of partial metric space in which the self-distance of any point of space may not be zero. The concept of a D-metric space introduced by the first author in [10]. Some specific examples of a D-metric space are presented in Dhage [11]. The details of a D-metric space and its topological properties appear in Dhage [12]. In [13], Dhage has proved some common fixed-point theorems for coincidentally commuting single-valued mappings in D-metric spaces satisfying a condition of generalized contraction. In [20], Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. Jungck and Rhoades [21, 22] defined the concepts of δ -compatible and weakly compatible mappings which extend the concept of compatible mappings in the single-valued setting to set-valued mappings.

Definition 1.1 ([32]). Let g and f be self-mappings of a set Y . A point $\varsigma \in Y$ is called a coincidence point of g and f iff $g\varsigma = f\varsigma$. In this case, $v = g\varsigma = f\varsigma$ is called a point of coincidence of g and f .

Definition 1.2 ([20]) Two single-valued mappings f and g of a metric space (Y, d) into itself are compatible if $\lim_{n \rightarrow \infty} d(fg\sigma_n, gf\sigma_n) = 0$ whenever $\{\sigma_n\}$ is a sequence in Y such that $\lim_{n \rightarrow \infty} f\sigma_n = \lim_{n \rightarrow \infty} g\sigma_n = t$ for some t in Y .

Definition 1.3 ([23]). Two self-mappings g and f of a metric space (Y, d) are said to be weakly compatible iff there is a point $\varsigma \in Y$ which is a coincidence point of g and f at which g and f commute; that is, $gf\varsigma = fg\varsigma$.

Proposition 1.1 ([2]) Let g and f be weakly compatible selfmaps of a nonempty set Y . If g and f have a unique point of coincidence $\varsigma = g\sigma = f\sigma$, then ς is the unique common fixed point of g and f .

Theorem 1.1 ([36]). Let (Y, d) be a complete metric space and f a self mapping on Y . If f is surjective and satisfies

$$d(f\sigma, f\varsigma) \geq \gamma d(\sigma, \varsigma) \quad (1.1)$$

for all $\sigma, \varsigma \in Y$, with $\gamma > 1$ then f has a unique fixed point in Y .

Sedghi et al. [31] introduced a new generalized metric space called an S -metric space. The S -metric space is a space with three dimensions. Sedghi et al. [31] asserted that every G -metric is an S -metric, see [31, Remarks 1.3 and 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [15] shows that this assertion is not correct. Moreover, the class of all S -metrics and the class of all G -metrics are distinct. Souayah et al. [35] have introduced S_b -metric space and established some fixed point theorems. Abbas et al. [1] introduced the notion of A -metric space, which generalization of the S -metric space.

In this work, we introduce a fixed point theorem for expansive mapping as a new tool in A -metric spaces. The obtained results generalize some facts in the literature.

2 Mathematical Preliminaries

In this part, some useful notions and facts will be given. In 2015, Abbas et al. [3] introduced the notion of A -metric space.

Definition 2.1 (see [3]) Let Y be a nonempty set. A mapping $A: Y^n \rightarrow [0, +\infty)$ is called an A -metric on Y if and only if for all $\sigma_i, a \in Y, i = 1, 2, 3, \dots, n$: the following conditions hold:

$$(A1). A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) \geq 0,$$

$$(A2). A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = 0 \text{ if and only if } \sigma_1 = \sigma_2 = \dots = \sigma_{n-1} = \sigma_n,$$

$$(A3). A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) \leq A(\sigma_1, \sigma_1, \sigma_1, \dots, (\sigma_1)_{n-1}, a) \\ + A(\sigma_2, \sigma_2, \sigma_2, \dots, (\sigma_2)_{n-1}, a) \\ + A(\sigma_3, \sigma_3, \sigma_3, \dots, (\sigma_3)_{n-1}, a) + \dots \\ + A(\sigma_{n-1}, \sigma_{n-1}, \sigma_{n-1}, \dots, (\sigma_{n-1})_{n-1}, a) \\ + A(\sigma_n, \sigma_n, \sigma_n, \dots, (\sigma_n)_{n-1}, a)]$$

The pair (Y, A) is called an A -metric space.

The following is the intuitive geometric example for A -metric spaces.

Example 2.2 (see [3]) Let $Y = [1, +\infty)$. Define $A: Y^n \rightarrow [0, +\infty)$ by

$$A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = \sum_{i=1}^n \sum_{i < j} |\sigma_i - \sigma_j|$$

for all $\sigma_i \in Y, i = 1, 2, \dots, n$.

Example 2.3 (see [3]) Let $Y = \mathbb{R}$. Define $A: Y^n \rightarrow [0, +\infty)$ by

$$\begin{aligned} A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) &= \left| \sum_{i=n}^2 \sigma_i - (n-1)\sigma_1 \right| \\ &+ \left| \sum_{i=n}^3 \sigma_i - (n-2)\sigma_2 \right| + \dots \\ &+ \left| \sum_{i=n}^{n-3} \sigma_i - 3\sigma_{n-3} \right| \\ &+ \left| \sum_{i=n}^{n-2} \sigma_i - 2\sigma_{n-2} \right| \\ &+ |\sigma_n - \sigma_{n-1}| \end{aligned}$$

for all $\sigma_i \in Y, i = 1, 2, \dots, n$.

Lemma 2.4 (see [3]) Let (Y, A) be an A -metric space. Then for all $\sigma, \varsigma \in Y$,

$$A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) = A(\varsigma, \varsigma, \varsigma, \varsigma, \dots, (\varsigma)_{n-1}, \sigma)$$

Lemma 2.5 (see [3]) Let (Y, A) be an A -metric space. Then for all $\sigma, \varsigma, \xi \in Y$,

$$\begin{aligned} A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \xi) &\leq (n-1)A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) \\ &+ A(\xi, \xi, \xi, \xi, \dots, (\xi)_{n-1}, \varsigma) \end{aligned}$$

and

$$\begin{aligned} A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \xi) &\leq (n-1)A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) \\ &+ A(\varsigma, \varsigma, \varsigma, \varsigma, \dots, (\varsigma)_{n-1}, \xi) \end{aligned}$$

Lemma 2.6 (see [3]) Let (Y, A) be an A -metric space. Then $(Y \times Y, D_A)$ is an A -metric space on $Y \times Y$, where D_A is given by for all $\sigma_i, \zeta_j \in Y, i, j = 1, 2, \dots, n$:

$$\begin{aligned} D_A((\sigma_1, \zeta_1), (\sigma_2, \zeta_2), (\sigma_3, \zeta_3), \dots, (\sigma_n, \zeta_n)) \\ = A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n) + A(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_n). \end{aligned}$$

Definition 2.7 (see [3]) Let (Y, A) be an A -metric space. Then

1. A sequence $\{\sigma_k\}$ is called convergent to σ in (Y, A) if

$$\lim_{k \rightarrow +\infty} A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma) \leq \epsilon$$

and we write $\lim_{k \rightarrow +\infty} \sigma_k = \sigma$.

2. A sequence $\{\sigma_k\}$ is called Cauchy in (Y, A) if

$$\lim_{k, m \rightarrow +\infty} A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have

$$A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma_m) \leq \epsilon.$$

3. (Y, A) is said to be complete if every Cauchy sequence in (Y, A) is a convergent.

Lemma 2.8 (see [3]) Let (Y, A) be an A -metric space. If the sequence $\{\sigma_k\}$ in Y converges to σ , then σ is unique.

Lemma 2.9 (see [3]) Every convergent sequence in A -metric space (Y, A) is a Cauchy sequence.

Lemma 2.10 ([3]) Let (Y, A) be an A -metric space. Then for all $\sigma, \zeta \in Y$,

$$A(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) = A(\zeta, \zeta, \zeta, \dots, \zeta, \sigma)$$

Lemma 2.11 ([3]) Let (Y, A) be an A -metric space. Then for all $\sigma, \zeta, \xi \in Y$,

$$A(\sigma, \sigma, \sigma, \dots, \sigma, \xi) \leq (n-1)A(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) + A(\xi, \xi, \xi, \dots, \xi, \zeta)$$

and

$$A(\sigma, \sigma, \sigma, \dots, \sigma, \xi) \leq (n-1)A(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) + A(\zeta, \zeta, \zeta, \dots, \zeta, \xi)$$

Note also that the following implications hold.

$$G\text{-metric space} \Rightarrow D^*\text{-metric space} \Rightarrow g\text{-metric space} \Rightarrow A\text{-metric space}$$

Definition 2.12 ([3]) The A -metric space (Y, A) is said to be bounded if there exists a constant $r > 0$ such that $A_b(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) \leq r$ for all $\sigma, \zeta \in Y$. Otherwise, Y is unbounded.

Definition 2.13 ([3]) Given a point σ_0 in A -metric space (Y, A) and a positive real number r , the set

$$B(\sigma_0, r) = \{\zeta \in Y : A(\zeta, \zeta, \zeta, \dots, \zeta, \sigma_0) < r\}$$

is called an open ball centered at σ_0 with radius r .

The set

$$\overline{B(\sigma_0, r)} = \{\zeta \in Y : A(\zeta, \zeta, \zeta, \dots, \zeta, \sigma_0) \leq r\}$$

is called a closed ball centered at σ_0 with radius r .

Definition 2.14 ([3]) A subset G in A -metric space (Y, A) is said to be an open set if for each $\sigma \in G$ there exists an $r > 0$ such that $B(\sigma, r) \subset G$. A subset $F \subset Y$ is called closed if $Y \setminus F$ is open.

Definition 2.15 ([3]) Let (Y, A) be an A -metric space with $s \geq 1$. A map $f: Y \rightarrow Y$ is said to be contraction if there exists a constant $\lambda \in [0, 1)$ such that

$$A(f\sigma_1, f\sigma_2, f\sigma_3, \dots, f\sigma_n) \leq \lambda A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n)$$

for all $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n \in Y$. In case

$$A(f\sigma_1, f\sigma_2, f\sigma_3, \dots, f\sigma_n) < A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n)$$

for all $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n \in Y, \sigma_i \neq \sigma_j$ for some $i \neq j, i, j \in \{1, 2, \dots, n\}$, f is called contractive mapping.

3. Main Results

First, we define following.

Definition 3.1 Let (Y, A) be an A -metric space with $s \geq 1$. A map $f: Y \rightarrow Y$ is said to be expansion mapping if there exists $\lambda > 1$ such that

$$A(f\sigma_1, f\sigma_2, f\sigma_3, \dots, f\sigma_n) \geq \lambda A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n)$$

for all $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n \in Y$. In case

$$A(f\sigma_1, f\sigma_2, f\sigma_3, \dots, f\sigma_n) > A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n)$$

for all $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n \in Y, \sigma_i \neq \sigma_j$ for some $i \neq j, i, j \in \{1, 2, \dots, n\}$, f is called expansive mapping.

Now, we give some fixed-point results for expansive mappings in a complete A -metric space. Our first main result as follows.

Theorem 3.1 Let (Y, A) be a complete A -metric space. Suppose the mappings $g, f: Y \rightarrow Y$ satisfy the condition

$$A\left(\underbrace{f\sigma, f\sigma, \dots, f\sigma}_{(n-1) \text{ times}}, f\zeta\right) \geq \alpha A\left(\underbrace{g\sigma, g\sigma, \dots, g\sigma}_{(n-1) \text{ times}}, g\zeta\right) + \beta A\left(\underbrace{f\sigma, f\sigma, \dots, f\sigma}_{(n-1) \text{ times}}, g\sigma\right) + \gamma A\left(\underbrace{f\zeta, f\zeta, \dots, f\zeta}_{(n-1) \text{ times}}, g\zeta\right) \tag{3.1}$$

for all $\sigma, \zeta \in Y$, where α, β, γ are nonnegative real numbers with $\alpha + \beta + \gamma > 1$. Assume the following hypotheses: (i) $\beta < 1$ and $\alpha \neq 0$, (ii) $g(Y) \subseteq f(Y)$, (iii) $f(Y)$ or $g(Y)$ is complete. Then f and g have a point of coincidence in Y . Moreover, if $\alpha > 1$, then the point of coincidence is

unique. If f and g are weakly compatible and $\alpha > 1$, then f and g have a unique common fixed point in Y .

Proof. Let $u_0 \in Y$ and choose $u_1 \in Y$ such that $gu_0 = fu_1$. This is possible since $g(Y) \subseteq f(Y)$. Continuing this process, we can construct a sequence $\{u_k\}_{k \in \mathbb{N}}$ in X such that $fu_k = gu_{k-1}$ for all $k \geq 1$.

By (3.1), we have

$$\begin{aligned} A\left(\underbrace{gu_{k-1}, gu_{k-1}, \dots, gu_{k-1}}_{(n-1) \text{ times}}, gu_k\right) &= A\left(\underbrace{fu_k, fu_k, \dots, fu_k}_{(n-1) \text{ times}}, fu_{k+1}\right) \\ &\geq \alpha A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, gu_{k+1}\right) \\ &\quad + \beta A\left(\underbrace{fu_k, fu_k, \dots, fu_k}_{(n-1) \text{ times}}, gu_k\right) \\ &\quad + \gamma A\left(\underbrace{fu_{k+1}, fu_{k+1}, \dots, fu_{k+1}}_{(n-1) \text{ times}}, gu_{k+1}\right) \\ \Rightarrow A\left(\underbrace{gu_{k-1}, gu_{k-1}, \dots, gu_{k-1}}_{(n-1) \text{ times}}, gu_k\right) &\geq \alpha A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, gu_{k+1}\right) \\ &\quad + \beta A\left(\underbrace{gu_{k-1}, gu_{k-1}, \dots, gu_{k-1}}_{(n-1) \text{ times}}, gu_k\right) \\ &\quad + \gamma A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, gu_{k+1}\right) \end{aligned}$$

The last inequality gives

$$A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, gu_{k+1}\right) \leq \lambda A\left(\underbrace{gu_{k-1}, gu_{k-1}, \dots, gu_{k-1}}_{(n-1) \text{ times}}, gu_k\right)$$

where $\lambda = \frac{1-\beta}{\alpha+\gamma}$. It is easy to see that $\lambda \in [0, 1)$. By induction, we get that

$$A\left(\underbrace{gu_k, gu_k, \dots, gu_k, gu_{k+1}}_{(n-1) \text{ times}}\right) \leq \lambda^k A\left(\underbrace{gu_0, gu_0, \dots, gu_0, gu_1}_{(n-1) \text{ times}}\right) \quad (3.2)$$

for all $k \geq 0$.

For $m, k \in \mathbb{N}$ with $m > k$, we have by repeated use of (Ab3)

$$\begin{aligned} A\left(\underbrace{gu_k, gu_k, \dots, gu_k, gu_m}_{(n-1) \text{ times}}\right) &\leq (n-1)A\left(\underbrace{gu_k, gu_k, \dots, gu_k, gu_{k+1}}_{(n-1) \text{ times}}\right) \\ &\quad + A\left(\underbrace{gu_m, gu_m, \dots, gu_m, gu_{k+1}}_{(n-1) \text{ times}}\right) \\ &= (n-1)A\left(\underbrace{gu_k, gu_k, \dots, gu_k, gu_{k+1}}_{(n-1) \text{ times}}\right) \\ &\quad + A\left(\underbrace{gu_{k+1}, gu_{k+1}, \dots, gu_{k+1}, gu_m}_{(n-1) \text{ times}}\right) \\ &\leq (n-1)A\left(\underbrace{gu_k, gu_k, \dots, gu_k, gu_{k+1}}_{(n-1) \text{ times}}\right) \\ &\quad + (n-1)A\left(\underbrace{gu_{k+1}, gu_{k+1}, \dots, gu_{k+1}, gu_{k+2}}_{(n-1) \text{ times}}\right) \\ &\quad + A\left(\underbrace{gu_m, gu_m, \dots, gu_m, gu_{k+2}}_{(n-1) \text{ times}}\right) \\ &\leq (n-1)A\left(\underbrace{gu_k, gu_k, \dots, gu_k, gu_{k+1}}_{(n-1) \text{ times}}\right) \\ &\quad + (n-1)A\left(\underbrace{gu_{k+1}, gu_{k+1}, \dots, gu_{k+1}, gu_{k+2}}_{(n-1) \text{ times}}\right) \end{aligned}$$

$$\begin{aligned}
 &+A\left(\underbrace{gu_{k+2}, gu_{k+2}, \dots, gu_{k+2}}_{(n-1) \text{ times}}, gu_m\right) \\
 &\leq (n-1)A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, gu_{k+1}\right) \\
 &+(n-1)A\left(\underbrace{gu_{k+1}, gu_{k+1}, \dots, gu_{k+1}}_{(n-1) \text{ times}}, gu_{k+2}\right) \\
 &+(n-1)A\left(\underbrace{gu_{k+2}, gu_{k+2}, \dots, gu_{k+2}}_{(n-1) \text{ times}}, gu_{k+3}\right) \\
 &+A\left(\underbrace{gu_m, gu_m, \dots, gu_m}_{(n-1) \text{ times}}, gu_{k+3}\right) \\
 &\leq (n-1)A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, gu_{k+1}\right) \\
 &+(n-1)A\left(\underbrace{gu_{k+1}, gu_{k+1}, \dots, gu_{k+1}}_{(n-1) \text{ times}}, gu_{k+2}\right) \\
 &+(n-1)A\left(\underbrace{gu_{k+2}, gu_{k+2}, \dots, gu_{k+2}}_{(n-1) \text{ times}}, gu_{k+3}\right) \\
 &+(n-1)A\left(\underbrace{gu_{k+3}, gu_{k+3}, \dots, gu_{k+3}}_{(n-1) \text{ times}}, gu_{k+4}\right) + \dots \\
 &+(n-1)A\left(\underbrace{gu_{m-2}, gu_{m-2}, \dots, gu_{m-2}}_{(n-1) \text{ times}}, gu_{m-1}\right) \\
 &+A\left(\underbrace{gu_{m-1}, gu_{m-1}, \dots, gu_{m-1}}_{(n-1) \text{ times}}, gu_m\right)
 \end{aligned}$$

$$\begin{aligned}
 &\leq (n - 1)[\lambda^k + \lambda^{k+1} + \dots + \lambda^{m-2} + \lambda^{m-1}]A\left(\underbrace{gu_0, gu_0, \dots, gu_0}_{(n-1) \text{ times}}, gu_1\right) \\
 &= (n - 1)\lambda^k[1 + \lambda + \lambda^2 + \dots + \lambda^{m-k-2} + \lambda^{m-k-1}]A\left(\underbrace{gu_0, gu_0, \dots, gu_0}_{(n-1) \text{ times}}, gu_1\right) \\
 &\leq (n - 1)\lambda^k[1 + \lambda + \lambda^2 + \dots]A\left(\underbrace{gu_0, gu_0, \dots, gu_0}_{(n-1) \text{ times}}, gu_1\right) \\
 &\leq (n - 1)\frac{\lambda^k}{1-\lambda}A\left(\underbrace{gu_0, gu_0, \dots, gu_0}_{(n-1) \text{ times}}, gu_1\right) \tag{3.3}
 \end{aligned}$$

Since $\lambda < 1, \lambda \in [0, 1)$. By taking limit as $k, m \rightarrow +\infty$ in above inequality, we get

$$\lim_{k,m \rightarrow +\infty} A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, gu_m\right) = 0.$$

for all $m > k$. Therefore, $\{gu_k\}$ is a Cauchy sequence in $g(Y)$. So $\{gu_k\}$ is a Cauchy sequence in $g(Y)$. Suppose that $g(Y)$ is a complete subspace of Y . Then there exists $v \in g(Y) \subseteq f(Y)$ such that $fu_k = gu_{k-1} \rightarrow v$. In case, $f(Y)$ is complete, this holds also with $v \in f(Y)$. Let $\xi \in Y$ be such that $f\xi = v$. By (3.1), we have

$$\begin{aligned}
 A\left(\underbrace{gu_{k-1}, gu_{k-1}, \dots, gu_{k-1}}_{(n-1) \text{ times}}, f\xi\right) &= A\left(\underbrace{fu_k, fu_k, \dots, fu_k}_{(n-1) \text{ times}}, f\xi\right) \\
 &\geq \alpha A\left(\underbrace{gu_k, gu_k, \dots, gu_k}_{(n-1) \text{ times}}, g\xi\right) \\
 &\quad + \beta A\left(\underbrace{fu_k, fu_k, \dots, fu_k}_{(n-1) \text{ times}}, gu_k\right) \\
 &\quad + \gamma A\left(\underbrace{f\xi, f\xi, \dots, f\xi}_{(n-1) \text{ times}}, g\xi\right)
 \end{aligned}$$

Taking $k \rightarrow \infty$ in the above inequality, we get

$$0 \geq (\alpha + \gamma)A\left(\underbrace{v, v, \dots, v}_{(n-1) \text{ times}}, g\xi\right)$$

This implies that $A\left(\underbrace{v, v, \dots, v}_{(n-1) \text{ times}}, g\xi\right) = 0$, since $\alpha + \gamma > 0$. Thus $g\xi = v$ and then $f\xi = g\xi = v$.

Therefore, v is a point of coincidence of g and f .

Now we suppose that $\alpha > 1$. Let ς be another point of coincidence of g and f . So $f\sigma = g\sigma = \varsigma$ for some $\sigma \in Y$. From (3.1), we have

$$\begin{aligned} A\left(\underbrace{v, v, \dots, v}_{(n-1) \text{ times}}, \varsigma\right) &= A\left(\underbrace{f\xi, f\xi, \dots, f\xi}_{(n-1) \text{ times}}, f\sigma\right) \\ &\geq \alpha A\left(\underbrace{g\xi, g\xi, \dots, g\xi}_{(n-1) \text{ times}}, g\sigma\right) \\ &\quad + \beta A\left(\underbrace{f\xi, f\xi, \dots, f\xi}_{(n-1) \text{ times}}, g\xi\right) \\ &\quad + \gamma A\left(\underbrace{f\sigma, f\sigma, \dots, f\sigma}_{(n-1) \text{ times}}, g\sigma\right) \\ &= \alpha A\left(\underbrace{v, v, \dots, v}_{(n-1) \text{ times}}, \varsigma\right) \end{aligned}$$

which implies that

$$(\alpha - 1)A\left(\underbrace{v, v, \dots, v}_{(n-1) \text{ times}}, \varsigma\right) \leq 0$$

Since $\alpha > 1$, we have $A\left(\underbrace{v, v, \dots, v}_{(n-1) \text{ times}}, \varsigma\right) = 0$ and hence $v = \varsigma$. Therefore, g and f have a unique point of coincidence in Y .

If g and f are weakly compatible, then by Proposition 1.1, g and f have a unique common fixed point in Y .

Corollary 2.1 Let (Y, A) be a complete A -metric space. Suppose the mappings $g, f: Y \rightarrow Y$ satisfy the condition

$$A\left(\underbrace{f\sigma, f\sigma, \dots, f\sigma, f\zeta}_{(n-1) \text{ times}}\right) \geq \alpha A\left(\underbrace{g\sigma, g\sigma, \dots, g\sigma, g\zeta}_{(n-1) \text{ times}}\right) \quad (3.4)$$

for all $\sigma, \zeta \in Y$, where $\alpha > 1$ is a constant. If $g(Y) \subseteq f(Y)$ and $f(Y)$ or $g(Y)$ is complete, then f and g have a unique point of coincidence in Y . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in Y .

Proof. It follows by taking $\beta = \gamma = 0$ in Theorem 3.1.

The following Corollary is the b-metric version of Banach's contraction principle.

Corollary 3.2 Let (Y, A) be a complete A -metric space. Suppose the mappings $g: Y \rightarrow Y$ satisfies the contractive condition

$$A\left(\underbrace{g\sigma, g\sigma, \dots, g\sigma, g\zeta}_{(n-1) \text{ times}}\right) \leq \lambda A\left(\underbrace{\sigma, \sigma, \dots, \sigma, \zeta}_{(n-1) \text{ times}}\right) \quad (3.5)$$

for all $\sigma, \zeta \in Y$, where $\lambda \in (0, 1)$ is a constant. Then g has a unique fixed point in Y . Furthermore, the iterative sequence $\{g^n\sigma\}$ converges to the fixed point.

Proof. It follows by taking $\beta = \gamma = 0$ and $f = I$, the identity mapping on Y , in Theorem 3.1.

Corollary 3.3 Let (Y, A) be a complete A -metric space. Suppose the mapping $f: Y \rightarrow Y$ is onto and satisfies

$$A\left(\underbrace{f\sigma, f\sigma, \dots, f\sigma, f\zeta}_{(n-1) \text{ times}}\right) \geq \alpha A\left(\underbrace{\sigma, \sigma, \dots, \sigma, \zeta}_{(n-1) \text{ times}}\right) \quad (3.6)$$

for all $\sigma, \zeta \in Y$, where $\alpha > 1$ is a constant. Then f has a unique fixed point in Y .

Proof. Taking $g = I$ and $\beta = \gamma = 0$ in Theorem 3.1, we obtain the desired result.

Corollary 3.4 Let (Y, A) be a complete A -metric space. Suppose the mapping $f: Y \rightarrow Y$ is onto and satisfies the condition

$$A\left(\underbrace{f\sigma, f\sigma, \dots, f\sigma, f\varsigma}_{(n-1) \text{ times}}\right) \geq \alpha A\left(\underbrace{\sigma, \sigma, \dots, \sigma, \varsigma}_{(n-1) \text{ times}}\right) + \beta A\left(\underbrace{f\sigma, f\sigma, \dots, f\sigma, \sigma}_{(n-1) \text{ times}}\right) + \gamma A\left(\underbrace{f\varsigma, f\varsigma, \dots, f\varsigma, \varsigma}_{(n-1) \text{ times}}\right) \quad (3.7)$$

for all $\sigma, \varsigma \in Y$, where α, β, γ are nonnegative real numbers with $\beta < 1$, $\alpha \neq 0$ $\alpha + \beta + \gamma > 1$. Then f has a fixed point in Y . Moreover, if $\alpha > 1$, then the fixed point of f is unique.

Proof. It follows by taking $g = I$ in Theorem 3.1.

We conclude with an example.

Example 3.1 Let $Y = [0, 1]$. Define $A: Y^n \rightarrow [0, +\infty)$ by

$$A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = \sum_{i=1}^n \sum_{i < j} |\sigma_i - \sigma_j|^2$$

for all $\sigma_i \in Y, i = 1, 2, \dots, n$. Therefore, (Y, A) is an A -metric space with $s = 2 > 1$.

Let us define $g, f: Y \rightarrow Y$ as

$$fu = \frac{u}{3} \text{ and } gu = \frac{u}{9} - \frac{u^2}{27}$$

for all $u \in Y$. Then, for every $\sigma, \varsigma \in Y$, the condition (3.1) holds for $\alpha = 9, \beta = \gamma = 0$. Thus, we have all the conditions of Theorem 3.1 and $0 \in Y$ is the unique common fixed point of f and g .

Conclusion

Wang et al. [36], proved some fixed point theorems for expansive mappings, which correspond to some contractive mappings in metric spaces. In the present article, we introduce a new approach to expansive mappings in fixed point theory by combining the ideas of Wang and establish a fixed

point theorem for expansive mappings as a new tool in A -metric spaces. The obtained results generalize some facts in the literature.

Conflict of Interest

No conflict of interest was declared by the authors.

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