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Common fixed-point theorem for a sequence of fuzzy mappings satisfying a rational contractive condition involving non-expansive mapping

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Abstract
In this article, we establish a common fixed-point theorem for a sequence of fuzzy mappings satisfying a rational contractive condition involving non-expansive mapping.

Keywords
Fuzzy sets, common fixed point, fuzzy mapping, non-expansive mapping.

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1. Introduction

The first important result on fixed points for contractive type mappings was the well-known Banach contraction principle [1] appeared in explicit form in Banach's thesis in 1922, where it was used and established the existence of a solution for an integral equation. Zadeh[2] familiarized the idea of a fuzzy set as a new way to represent vagueness in everyday life. The study of fixed point theorems in fuzzy mathematics was investigated by Weiss [3], Butnariu [4], Singh and Talwar [5], Mihet [6], Qiu et al. [7], and Beg and Abbas [8] and many others. Heilpern [9] first used the concept of fuzzy mappings to prove the Banach contraction principle for fuzzy mappings on a complete metric linear space. The result obtained by Heilpern [9] is a fuzzy analogue of the fixed point theorem for multivalued mappings of Nadler et al. [10]. Bose and Sahani [11], Vijayaraju and Marudai [12], improved the result of Heilpern. In some earlier work, Watson and Rhoades [13],[14] proved several fixed-point theorems involving a very general contractive definition. In this paper, we prove a common fixed point theorem for sequence of fuzzy mappings satisfyinga rational contractive condition involving nonexpansive mapping.Our results extend and generalized the correspondingresults of Bose and Sahani [11], Vijayaraju and Mohanraj [12] and Rhoades [15],[16], Salujaet al. [18] and Das and Gupta [19].

2. Preliminaries

We recall some mathematical basics and definitions to make this paper self-sufficient (see [9]).

Definition 2.1. *Let (M,m) be a complete linear metric space and $\mathscr{F}(M)$, the collection of all fuzzy sets in M. A fuzzy set in Mis a function with domain M and values in $[0,1]$. If Ais a fuzzy set and $\sigma \in M$, then the function value $A(\sigma)$ is called the grade of membership of σ in A. The α -level set of Ais denoted by*

$$A_\alpha = \{\sigma : A(\sigma) \geq \alpha\} \text{ if } \alpha \in (0,1]$$
$$A_0 = \overline{\{\sigma : A(\sigma) > 0\}}$$

where $\bar{B}$ stands for the (non-fuzzy) closure of a set B.

Definition 2.2. *A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex for each $\alpha \in (0,1]$ andsup $_{\sigma\in M}A(\sigma)=1$, when Ais an approximate quantity and $A(\sigma_0)=1$ for some $\sigma_0 \in M$, Ais identified with an approximation of σ_0. From the collection $\mathscr{F}(M)$, a sub-collection of all appropriate quantities is denoted as $\mathscr{W}(M)$.*

Definition 2.3. *The distance between two appropriate quantities is defined by the following scheme. Let $A,B \in \mathscr{W}(M)$ and $\alpha \in [0,1]$,*

$$D_\alpha(A,B) = \inf_{\sigma\in A_\alpha,\varsigma\in B_\alpha} m(\sigma,\zeta)$$
$$H_\alpha(A,B) = \operatorname{dist} m(A_\alpha,B_\alpha)$$
$$H(A,B) = \sup_\alpha D_\alpha(A,B)$$

wherein the dist is in the sense of Hausdorff distance .The function D_α is called an α -distance (induced by m), H_αaα -distance (induced by dist) and H a distance between Aand B. Note that D_α is a non-decreasing function of α.

Definition 2.4. *Let $A,B \in \mathscr{W}(M)$. Then Ais said to be more accurate than B, denoted by $A \subset B$, iff $A(\sigma) \le B(\sigma)$ for each $\sigma \in M$. The relation $\subset$ induces a partial ordering on the family $\mathscr{W}(M)$.*

Definition 2.5. *Let Y be an arbitrary set and M be any metric space. Sis called a fuzzy mapping if and only if S is a mapping from the set Y into $\mathscr{W}(M)$. A fuzzy mapping F is a fuzzy subset of $Y \times M$ with membership function $S(\varsigma,\sigma)$. The function value $S(\zeta,\sigma)$ is the grade of membership of σ in $S(S)$. Note that each fuzzy mapping is a set valued mapping. Let $A \in S(M), B \in S(Y)$. Then fuzzy set $S(A)$ in $S(M)$ is defined by*

$$S(A)(\sigma) = \sup_{\varsigma\in M}(S(\varsigma,\sigma)\bigwedge A(\varsigma)), \sigma \in M$$

and the fuzzy set $S^{-1}(B)$ in $S(Y)$ is defined by

$$S^{-1}(B)(\varsigma) = \sup_{\sigma\in M} S(\varsigma,\sigma)\bigwedge B(\sigma), \varsigma \in \mathrm{Y}$$

Lee [17] proved the following.

Lemma 2.6. *Let (M,m) be a complete linear metric space, S is a fuzzy mapping from Minto $\mathscr{W}(M)$ and $\sigma_0 \in M$, then there exists an $\sigma_1 \in M$ such that $\{\sigma_1\} \subset S(\sigma_0)$.*

The following two lemmas are due to Heilpern [9].

Lemma 2.7. *Let $\sigma \in M, A \in \mathscr{W}(M)$ and $\{\sigma\}$ a fuzzy set with membership function equal to a characteristic function of $\{\sigma\}$. If $\{\sigma\} \subset A$, then $D_\alpha(\sigma,A)=0$ for each $\alpha \in [0,1]$.*

Lemma 2.8. *Let $A,B \in \mathscr{W}(M), \alpha \in [0,1]$ and $D_\alpha(A,B) = \inf_{\sigma\in A_\alpha,\varsigma\in B_\alpha} m(\sigma,\zeta)$,where $A_\alpha = \{\sigma : A(\sigma) \ge \alpha\}$, then*

$$D_\alpha(\sigma,A) \le m(\sigma,\varsigma) + D_\alpha(\zeta,A)$$

for each $\sigma,\varsigma \in M$.

Lemma 2.9. *Let $H_\alpha(A,B) = \operatorname{dist} m(A_\alpha,B_\alpha)$,where 'dist' isthe Hausdorff distance. If $\{\sigma_0 \subset A\}$, then $D_\alpha(\sigma_0,B) \le H_\alpha(A,B)$ for each $B \in \mathscr{W}(M)$.*

Rhoades [15] proved the following common fixed point theorem involving a very general contractive condition, for fuzzy mappings on complete linear metric space. He proved the following theorem.

Theorem 2.10. *Let (M,m) be a complete linear metric space and let S,T be fuzzy mappings from Minto $\mathscr{W}(M)$ satisfying*

$$H(S\sigma,Tc) \le Q(m(\sigma,\varsigma)) \tag{2.1}$$

where

$$m(\sigma,\varsigma) = \max\left\{m(\sigma,\varsigma), D_\alpha(\sigma,S\sigma), D_\alpha(\varsigma,T\varsigma), \frac{D_\alpha(\sigma,T\varsigma)+D_\alpha(s,S\sigma)}{2}\right\}$$

and Q is a real-valued function defined on D, the closure of the range of m, satisfying the following three conditions:

(a) $0 < Q(s) < s$for each $s \in D\backslash\{0\}$ and $Q(0)=0$,

(b) Q is non-decreasing on D, and

(c) $f(s) = s/s - Q(s)$ is non-increasing on $D\backslash\{0\}$.

Then there exists a point z in X such that $\{z\} \subset Sz \cap Tz$.

In [16] Rhoades, generalized the result of Theorem 2.10 for sequence of fuzzy mappings on complete linear metric space. He proved the following theorem.

Theorem 2.11. *Let f be a non-expansive self-mapping of a complete linear metric space (M,m) and $\{S_i\}$bea sequence of fuzzy mappings from M into $W(M)$. For each pair of fuzzy mappings S_i,S_j and for any $\sigma \in M, \{\eta_\sigma\} \subset S_i(\sigma)$, there exists a $\{\mu_\varsigma\} \subset S_j(s)$ for all $\varsigma \in M$ such that*

$$D(\{\eta_\sigma\},\{\mu_\varsigma\}) \le Q(m(\sigma,\zeta)) \tag{2.2}$$

Where

$$m(\sigma,\varsigma) = \max\left\{(f(\sigma),f(s)), m(f(\sigma),f(\eta_\sigma)), m(f(s),f(\mu_\varsigma)), \frac{m(f(\sigma),f(\mu_\varsigma)), +m(f(\varsigma),f(\eta_\sigma))}{2}\right\}$$

and Q satisfying the conditions (a)-(c) of Theorem 2.10. Then there exists $\{z\} \subset \bigcap_{i=1}^{\infty} S_i(z)$.

3. Main Results

Now, we give our first main result.

Theorem 3.1. *Let f be a non-expansive self-mapping of a complete linear metric space (M,m) and $\{S_i\}$bea sequence of fuzzy mappings from M into $W(M)$. For each pair of fuzzy mappings S_i, S_j and for any $\sigma \in M, \{\eta_\sigma\} \subset S_i(\sigma)$, there exists a $\{\mu_\varsigma\} \subset S_j(s)$ for all $\varsigma \in M$ such that*

$$D(\{\eta_\sigma\},\{\mu_\varsigma\}) \leq Q\Big(\max\Big\{m(f(\sigma),f(s)), m(f(\sigma),f(\eta_\sigma)), m(f(s),f(\mu_\varsigma)), \frac{m(f(\varsigma),f(\mu_c))[1+m(f(\sigma),f(\eta_\sigma))]}{1+m(f(\sigma),f(s))}\Big\}\Big) \tag{3.1}$$

and Q satisfying the conditions (a)-(c) of Theorem 2.10. Then there exists $\{\xi\} \subset \bigcap_{i=1}^{\infty} S_i(\xi)$.

Proof. Let $\sigma_0 \in M$. Then we can choose $\sigma_1 \in M$ such that $\{\sigma_1\} \subset S\sigma_0$ by Lemma 2.6. From the hypothesis, there exists an $\sigma_1 \in M$ such that $\{\sigma_2\} \subset S\sigma_1$ and Since f is a nonexpansive self- mapping, from (3.1), we have

$$\begin{aligned}
&D(\{\sigma_1\},\{\sigma_2\})\\
&\leq Q\left(\max\left\{\begin{array}{c} m(f(\sigma_0),f(\sigma_1)), m(f(\sigma_1),f(\sigma_2)),\\ m(f(\sigma_0),f(\sigma_1)),\\ \frac{m(f(\sigma_1),f(\sigma_2))[1+m(f(\sigma_0),f(\sigma_1))]}{1+m(f(\sigma_0),f(\sigma_1))}\end{array}\right\}\right)\\
&\leq \max\{m(f(\sigma_0),f(\sigma_1)), m(f(\sigma_1),f(\sigma_2))\}
\end{aligned}$$

$$\leq \max\{m(\sigma_0,\sigma_1), m(\sigma_1,\sigma_2)\} \tag{3.2}$$

Inductively, we obtain a sequence $\{\sigma_n\}$ such that $\{\sigma_{n+1}\} \subset S_{n+1}(\sigma_n)$ and

$$\begin{aligned}
&D(\{\sigma_n\},\{\sigma_{n+1}\})\\
&\leq Q\left(\max\left\{\begin{array}{c} m(f(\sigma_{n-1}),f(\sigma_n)), m(f(\sigma_n),f(\sigma_{n+1})),\\ m(f(\sigma_{n-1}),f(\sigma_n)),\\ \frac{m(f(\sigma_n),f(\sigma_{n+1}))[1+m(f(\sigma_{n-1}),f(\sigma_n))]}{1+m(f(\sigma_{n-1}),f(\sigma_n))}\end{array}\right\}\right)\\
&\leq \max\{m(f(\sigma_{n-1}),f(\sigma_n)), m(f(\sigma_n),f(\sigma_{n+1}))\}
\end{aligned}$$

$$\leq \max\{m(\sigma_{n-1},\sigma_n), m(\sigma_n,\sigma_{n+1})\} \tag{3.3}$$

Since $D(\{\sigma_n\},\{\sigma_{n+1}\}) = m(\sigma_n,\sigma_{n+1})$ it follows from (3.2) thatm $(\sigma_n,\sigma_{n+1}) < m(\sigma_{n-1},\sigma_n)$. Using this fact back in (3.1), we obtain that $m(\sigma_n,\sigma_{n+1}) \leq m(\sigma_{n-1},\sigma_n)$. Substituting into (3.2) we obtain

$$\begin{aligned}
m(\sigma_n,\sigma_{n+1}) &< Q(m(\sigma_{n-1},\sigma_n)) < Q^2(m(\sigma_{n-2},\sigma_{n-1}))\\
&< \cdots < Q^n(m(\sigma_0,\sigma_1))
\end{aligned} \tag{3.4}$$

From Lemma 2 of [17], $\lim_{n\to\infty} Q^n(m(\sigma_0,\sigma_1)) = 0$. To show that $\{\sigma_n\}$ is Cauchy, choose N so large that $Q^n(m(\sigma_0,\sigma_1)) \leq \left(\frac{1}{2}\right)^n$ for all $n > N$. Then, for $r > n > N$

$$\begin{aligned}
m(\sigma_n,\sigma_r) &\leq m(\sigma_n,\sigma_{n+1}) + m(\sigma_{n+1},\sigma_{n+2}) + \cdots\\
&\quad + m(\sigma_{r-1},\sigma_r)\\
&= \sum_{j=n}^{r-1} m(\sigma_j,\sigma_{j+1}) \leq \sum_{j=n}^{r-1} Q^j(m(\sigma_0,\sigma_1))\\
&\leq \sum_{j=n}^{r-1}\left(\frac{1}{2}\right)^j < \left(\frac{1}{2}\right)^{n-1}
\end{aligned} \tag{3.5}$$

and $\{\sigma_n\}$ is Cauchy, hence convergent. Call the limit ξ. Let S_m be an arbitrary member of the sequence$\{S_i\}$. Since $\{\sigma_n\} \subset S_r(\sigma_{n-1})$, there exists a $\mu_n \in M$ such that $\{\mu_n\} \subset S_r(\xi)$ for all n and applying (3.1), we have

$$\begin{aligned}
&D(\{\sigma_n\},\{\{\mu_n\}\})\\
&\leq Q\left(\max\left\{\begin{array}{c} m(f(\sigma_{n-1}),f(\xi)), m(f(\xi),f(\mu_n)),\\ m(f(\sigma_{n-1}),f(\sigma_n)),\\ \frac{m(f(\xi),f(\mu_n))[1+m(f(\sigma_{n-1}),f(\sigma_n))]}{1+m(f(\sigma_{n-1}),f(\xi))}\end{array}\right\}\right)\\
&< Q\left(\max\left\{\begin{array}{c} m(f(\sigma_{n-1}),f(\xi)), m(f(\xi),f(\mu_n)),\\ m(f(\sigma_{n-1}),f(\sigma_n)),\\ \frac{m(f(\xi),f(\mu_n))[1+m(f(\sigma_{n-1}),f(\sigma_n))]}{1+m(f(\sigma_{n-1}),f(\xi))}\end{array}\right\}\right)
\end{aligned}$$

$$\leq Q\left(\max\left\{m(\sigma_{n-1},\xi), \frac{m(\xi,\mu_n)[1+m(\sigma_{n-1},\sigma_n)]}{1+m(\sigma_{n-1},\xi)}\right\}\right) \tag{3.6}$$

Suppose that $\lim_{n\to\infty}\mu_n \neq \xi$. Taking the limit as $n \to \infty$ yields, since Q is continuous (Lemma 1 of [13]

$$\limsup_{n\to\infty} m(\xi,\mu_n) \leq Q\left(\limsup_{n\to\infty} m(\xi,\mu_n)\right) < \limsup_{n\to\infty} m(\xi,\mu_n)$$

This is a contradiction. Therefore,lim $_{n\to\infty}\mu_n = \xi$. Since $S_r(\xi) \in W(M), S_r(\xi)$ is upper semi continuous and therefore, $limsup_{n\to\infty}[S_r(\xi)](\mu_n) \leq [S_r(\xi)](\xi)$. Since $\{\mu_n\} \subset S_r(\xi)$ for all $[S_r(\xi)](\xi) = 1$. Hence $\{\xi\} \subset S_r(\xi)$. Since S_r is arbitrary, $\{\xi\} \subset \cap_{i=1}^{\infty} S_i(\xi)$. □

Theorem 3.2. *Let f be a nonexpansive self-mapping of a complete linear metric space (M,m) and $\{S_i\}$ be a sequence of fuzzy mappings from M into $W(M)$. For each pair of fuzzy mappings S_i, S_j and for any $\sigma \in M, \{\eta_\sigma\} \subset S_i(\sigma)$, there exists a $\{\mu_\varsigma\} \subset S_j(s)$ for all $s \in M$ such that*

$$\begin{aligned}
&D(\{\eta_\sigma\},\{\mu_\varsigma\})\\
&\leq \max\left\{\begin{array}{c} m(f(\sigma),f(s)), m(f(s),f(\mu_\varsigma)),\\ m(f(\sigma),f(\eta_\sigma)),\\ \frac{m(f(s),f(\mu_\varsigma))[1+m(f(\sigma),f(\eta_\sigma))]}{1+m(f(\sigma),f(s))}\end{array}\right\}\\
&\quad - \mathrm{w}\left(\begin{array}{c} m(f(\sigma),f(\varsigma)), m(f(s),f(\mu_\varsigma)),\\ m(f(\sigma),f(\eta_\sigma)),\\ \frac{m(f(s),f(\mu_s))[1+m(f(\sigma),f(\eta_\sigma))]}{1+m(f(\sigma),f(s))}\end{array}\right)
\end{aligned} \tag{3.7}$$

for all $\sigma, \varsigma \in M, w: R^+ \to R^+$ *be a continuous functionsuch that* $0 < w(r) < r$ *for all* $r > 0$. *Then there exists* $\{\xi\} \subset \bigcap_{i=1}^{\infty} S_i(\xi)$*, i.e.* ξ *is a commonfixed point of the sequence of fuzzy mappings.*

Proof. Let $\sigma_0 \in M$. Then we can choose $\sigma_1 \in M$ such that $\{\sigma_1\} \subset S\sigma_0$ by Lemma 2.6. From the hypothesis, there exists an $\sigma_1 \in M$ such that $\{\sigma_2\} \subset S\sigma_1$ and Since f is a non-expansive self-mapping, from (3.7), we have

$$\begin{aligned}
&D(\{\sigma_1\},\{\sigma_2\}) \\
&\leq \max\left\{\begin{array}{c} m(f(\sigma_0),f(\sigma_1)), m(f(\sigma_1),f(\sigma_2)), \\ m(f(\sigma_0),f(\sigma_1)), \\ \frac{m(f(\sigma_1),f(\sigma_2))[1+m(f(\sigma_0),f(\sigma_1))]}{1+m(f(\sigma_0),f(\sigma_1))} \end{array}\right\} \\
&\quad - \mathrm{w}\left(\max\left\{\begin{array}{c} m(f(\sigma_0),f(\sigma_1)), m(f(\sigma_1),f(\sigma_2)), \\ m(f(\sigma_0),f(\sigma_1)), \\ \frac{m(f(\sigma_1),f(\sigma_2))[1+m(f(\sigma_0),f(\sigma_1))]}{1+m(f(\sigma_0),f(\sigma_1))} \end{array}\right\}\right) \\
&= \max\{m(f(\sigma_0),f(\sigma_1)), m(f(\sigma_1),f(\sigma_2))\} \\
&\quad - \mathrm{w}(\max\{m(f(\sigma_0),f(\sigma_1)), m(f(\sigma_1),f(\sigma_2))\}) \\
&\leq \max\{m(\sigma_0,\sigma_1), m(\sigma_1,\sigma_2)\} \\
&\quad - \mathrm{w}(\max\{m(\sigma_0,\sigma_1), m(\sigma_1,\sigma_2)\})
\end{aligned}$$

The last inequality gives

$$\begin{aligned}
m(\sigma_1,\sigma_2) =& D(\{\sigma_1\},\{\sigma_2\}) \\
&\leq \max\{m(\sigma_0,\sigma_1), m(\sigma_1,\sigma_2)\} \\
&\quad - \mathrm{w}(\max\{m(\sigma_0,\sigma_1), m(\sigma_1,\sigma_2)\})
\end{aligned}$$

which implies that

$$m(\sigma_1,\sigma_2) \leq m(\sigma_0,\sigma_1) - \mathrm{w}(m(\sigma_0,\sigma_1)) \tag{3.8}$$

Similarly

$$m(\sigma_2,\sigma_3) \leq m(\sigma_1,\sigma_2) - \mathrm{w}(m(\sigma_1,\sigma_2)) \tag{3.9}$$

Inductively, we obtain a sequence $\{\sigma_n\}$ such that $\{\sigma_{n+1}\} \subset S_{n+1}(\sigma_n)$ and

$$m(\sigma_n,\sigma_{n+1}) \leq m(\sigma_{n-1},\sigma_n) - \mathrm{w}(m(\sigma_{n-1},\sigma_n)) \tag{3.10}$$

Adding $(3.8)-(3.10)$, we obtain

$$\sum_{i=0}^{n} \mathrm{w}(m(\sigma_i,\sigma_{i+1})) \leq m(\sigma_0,\sigma_1) - m(\sigma_n,\sigma_{n+1}) < m(\sigma_0,\sigma_1)$$

Therefore

$$\sum_{i=0}^{n} \mathrm{w}(m(\sigma_i,\sigma_{i+1})) < \infty, \lim_{n\to\infty} \mathrm{w}(m(\sigma_n,\sigma_{n+1})) = 0$$

Now suppose that $\{\sigma_n\}$ is not a Cauchy sequence, thenthere is an $\varepsilon > 0$ such that for each positive even integer $2k$, there exists positive even integer $2r > 2n > 2k$ such that

$$m(\sigma_{2n},\sigma_{2r}) \geq \varepsilon \tag{3.11}$$

Also, for each $2k$, we may find the least $2m$ exceeding $2n$ such that

$$m(\sigma_{2n},\sigma_{2r-2}) < \varepsilon \tag{3.12}$$

Since $\{m(\sigma_n,\sigma_{n+1})\}$ is a decreasing sequence of non-negative terms, it converges, call the limit ξ. Suppose that $\xi > 0$. Then, since w is continuous,

$$\lim_{n\to\infty} \mathrm{w}(m(\sigma_n,\sigma_{n+1})) = w(\xi)$$

But $\lim_{n\to\infty} \mathrm{w}(m(\sigma_n,\sigma_{n+1})) = 0$. Hence $w(\xi) = 0$, which is a contradiction to the fact that $0 < w(\xi) < \xi$. Hence $\xi = 0$ and then

$$\lim_{n\to\infty} m(\sigma_n,\sigma_{n+1}) = 0 \tag{3.13}$$

Now

$$\begin{aligned}
\varepsilon \leq& m(\sigma_{2n},\sigma_{2r}) \leq m(\sigma_{2n},\sigma_{2r-2}) + m(\sigma_{2r-2},\sigma_{2r-1}) \\
&+ m(\sigma_{2r-1},\sigma_{2r})
\end{aligned} \tag{3.14}$$

Using (3.11)-(3.14), we obtain

$$m(\sigma_{2n},\sigma_{2r}) \to \varepsilon \text{ as } k \to \infty \tag{3.15}$$

Note that

$$\begin{aligned}
|m(\sigma_{2r},\sigma_{2n+1}) - m(\sigma_{2r},\sigma_{2n})| &\leq m(\sigma_{2n},\sigma_{2n+1}) \\
|m(\sigma_{2r+1},\sigma_{2n+1}) - m(\sigma_{2r},\sigma_{2n+1})| &\leq m(\sigma_{2r},\sigma_{2r+1}) \\
|m(\sigma_{2r},\sigma_{2n+2}) - m(\sigma_{2r},\sigma_{2n+1})| &\leq m(\sigma_{2n+1},\sigma_{2n+2}) \\
|m(\sigma_{2r+1},\sigma_{2n+2}) - m(\sigma_{2r+1},\sigma_{2n+1})| &\leq m(\sigma_{2n+1},\sigma_{2n+2})
\end{aligned}$$

which implies thatas $k \to \infty$,

$$m(\sigma_{2r},\sigma_{2n+1}) \to \varepsilon, \quad m(\sigma_{2r+1},\sigma_{2n+1}) \to \varepsilon$$

$$m(\sigma_{2r},\sigma_{2n+2}) \to \varepsilon, m(\sigma_{2r+1},\sigma_{2n+2}) \to \varepsilon \tag{3.16}$$

Again applying (3.7), we get

$$\begin{aligned}
&m(\sigma_{2r+1},\sigma_{2n+2}) = D(\{\sigma_{2r+1}\},\{\sigma_{2n+2}\}) \\
&\leq \max\left\{\begin{array}{c} m(f(\sigma_{2r}),f(\sigma_{2n+1})), m(f(\sigma_{2n+1}), \\ f(\sigma_{2n+2})), m(f(\sigma_{2m}),f(\sigma_{2m+1})) \\ \frac{m(f(\sigma_{2n+1}),f(\sigma_{2n+2}))[1+m(f(\sigma_{2m}),f(\sigma_{2m+1}))]}{1+m(f(\sigma_{2m}),f(\sigma_{2n+1}))} \end{array}\right\} \\
&- \mathrm{w}\left(\begin{array}{c} m(f(\sigma_{2r}),f(\sigma_{2n+1})), m(f(\sigma_{2n+1}),f(\sigma_{2n+2})), \\ m(f(\sigma_{2r}),f(\sigma_{2r+1})) \\ \frac{m(f(\sigma_{2n+1}),f(\sigma_{2n+2})[1+m(f(\sigma_{2m}),f(\sigma_{2m+1}))])}{1+m(f(\sigma_{2m}),f(\sigma_{2n+1}))} \end{array}\right) \\
&\leq \max\left\{\begin{array}{c} m(\sigma_{2r},\sigma_{2n+1}), m(\sigma_{2n+1},\sigma_{2n+2}), \\ m(\sigma_{2r},\sigma_{2r+1}), \\ \frac{m(\sigma_{2n+1},\sigma_{2n+2})[1+m(\sigma_{2r},\sigma_{2r+1})]}{1+m(\sigma_{2r},\sigma_{2n+1})} \end{array}\right\} \\
&- \mathrm{w}\left(\max\left\{\begin{array}{c} m(\sigma_{2r},\sigma_{2n+1}), m(\sigma_{2n+1},\sigma_{2n+2}), \\ m(\sigma_{2r},\sigma_{2r+1}), \\ \frac{m(\sigma_{2n+1},\sigma_{2n+2})[1+m(\sigma_{2r},\sigma_{2r+1})]}{1+m(\sigma_{2r},\sigma_{2n+1})} \end{array}\right\}\right)
\end{aligned}$$

Using (3.13), (3.16) and taking the limit as $k \to \infty$, we get

$$\varepsilon \leq \max\{\varepsilon, 0, 0\} - \mathrm{w}(\max\{\varepsilon, 0, 0\})$$

which gives a contradiction. Thus $\{\sigma_n\}$ is a Cauchysequence and since M is complete, it converges to some $\xi \in M$.

Let S_r be an arbitrary member of the sequence $\{S_i\}$. Since $\{\sigma_n\} \subset S_r(\sigma_{n-1})$, by Lemma 2.6, there exists a $\mu_n \in M$ such that $\{\mu_n\} \subset S_r(\xi)$ for all n and applying (3.7) again, we have

$$\begin{aligned}
m(\sigma_n, \mu_n) &= D(\{\sigma_n\}, \{\mu_n\}) \\
&\leq \max\Big\{m(f(\sigma_{n-1}), f(\xi)), \\
&\quad \frac{m(f(\xi), f(\mu_n))[1 + m(f(\sigma_{n-1}), f(\sigma_n))]}{1 + m(f(\sigma_{n-1}), f(\xi))}\Big\} \\
&\quad - \mathrm{w}\Big(\max\Big\{m(f(\sigma_{n-1}), f(\xi)), \\
&\quad \frac{m(f(\xi), f(\mu_n))[1 + m(f(\sigma_{n-1}), f(\sigma_n))]}{1 + m(f(\sigma_{n-1}), f(\xi))}\Big\}\Big) \\
&\leq \max\left\{m(\sigma_{n-1}, \xi), \frac{m(\xi, \mu_n)[1 + m(\sigma_{n-1}, \sigma_n)]}{1 + m(\sigma_{n-1}, \xi)}\right\} \\
&\quad - \mathrm{w}\left(\max\left\{m(\sigma_{n-1}, \xi), \frac{m(\xi, \mu_n)[1 + m(\sigma_{n-1}, \sigma_n)]}{1 + m(\sigma_{n-1}, \xi)}\right\}\right)
\end{aligned}$$

Suppose that $\lim_{n\to\infty} \mu_n \neq \xi$. Taking the limit as $n \to \infty$ yields

$$m(\xi, \mu_n) \leq m(\xi, \mu_n) - \mathrm{w}(m(\xi, \mu_n))$$

Since w is continuous, we get a contradiction. Therefore, $\lim_{n\to\infty} \mu_n = \xi$. Hence $\{\xi\} \subset S_r(\xi)$. Since S_r is arbitrary, $\{\xi\} \subset \bigcap_{i=1}^{\infty} S_i(\xi)$. □

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