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Numerical Treatment of Fourth Order Self-Adjoint Singularly Perturbed Boundary Value Problems via Septic B-Spline Method

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Abstract- In this communication, A septic B-spline method (SBSM) is descrived for numerical treatment of fourth order self adjoint (FOSA) singularly perturbed boundary value problems (SPBVPs) and method is directly implemented on the problems without decreasing the order of the original differential equations. Convergence of the SBSM is proved and found that it gives 4[th] order convergence results. The present technique is applied on two numerical problems which supports the theoretical proofs .

***Keywords*:** Septic B-spline method, Singularly perturbed boundary value problems, Fourth order self-adjoint, Uniform Convergence.

Mathematics Subject Classification (MSC)2010: 65L11

1. Introduction

We consider the following fourth order self adjoint SPBVPs:

$$-\varepsilon u^{iv}(y)+a(y)u(y)=r(y), \qquad y\in[p,\ q] \tag{1}$$

with the boundary condition (BC):

$$\begin{aligned} y(p)&=\alpha_1, & y(q)&=\beta_1,\\ y'(p)&=\alpha_2, & y'(q)&=\beta_2. \end{aligned} \tag{2}$$

where α_1, α_2, β_1 and β_2 are constants and perturbation parameter ε is $0<\varepsilon\ll 1$. We suppose that the functions $a(y)$ and $r(y)$ are smooth functions in $[p,q]$. A singular perturbation problems (SPPs) is arise in nuomarus regions of mathematical and engineering science for instant fluid dynamics, optimal control theory, chemistry, hydrodynamics, quantum physics, chemical reactor theory and reaction-diffusion process etc.

Beginning in the mid of the nineteenth century and especially during the past few decades, there have been intense efforts in numerical solution of SPPs with a huge literature. Researchers were interested in the development of varies numerical techniques that work for all estimations of the singular perturbation parameter ε. Numerical treatment of SPPs presents some computational challenges because of boundary layers regions. There are sone more possible way to tackle these problems, for the detail of the methods researcher may refer the books [4, 11] references therein. Wang [16] has presented the numerical solution of nonlinear SPP on non uniform nodal points which are more compact in the inner boundary layer as compared the outer boundary layer. The basic concept of the numerical method is based upon the integral equation and found that it gives fourth order uniform accuracy. Vulanovic [15] has studied the transformation discretization method and which have a relation of layer–resolving (LR) transformations with mesh generating functions to evaluate the approximate solution of non linear SPBVPs. The examination is done for 1-D quasilinear problem in absence of turning point, which are discretized through first order finite difference method (FDM). The approximate solutions of 1-D SPPs converge uniformly in ε when general LR function has been used to develop the discretization mesh points. Mishra and Saini [13] have studied the various numerical technique to find the approximate solution of SPBVPs.

Cakir et al. [5] have developed the FDM on a uniform mesh to find the numerical treatment of SP three points BVPs. They have also discussed the nature of exact solution at boundary points and derived the first derivative. Mohapatra and Shakti [10] have presented an adaptive mesh to obtain the approximate treatment of one dimensional singularly perturbed pseudo parabolic problem. Backward Euler method is used to obtain the mesh point of time derivative and spatial derivative is discretized with central difference method on uniform mesh. Avijit and Natesan [3] have described the convergence analysis of streamline-diffusion finite element method (FEM) for two-parameter SPBVPs in the discrete streamline-diffusion norm. Another very interesting and different boundary value problems is solved by using spline techniques [8, 14] and for the details of spline function sereachers may refer the book [11]. Akram and Naheed [1, 2] have described the approximate solution of fourth order SPBVPs by using quintic and septic spline in very effective manner. Saini and Mishra [13] have presented the quartic B-spline method for approximate treatment of 3rd order self adjoint SPBVPs. Lodhi and Mishra [7] have developed non polynomial spline for numerical treatment of fourth order SPBVPs. In this article, we developed a SBSM for numerical treatment of FOSA SPBVPs. Remeaning part of the article is organized as follows: description of the SBSM is presented in section 2 and section 3 is devoted for the convergence of the SBSM. Numerical examples are discussed in section 4 and conclusion of the work is described in the section 5.

2. Description of Septic B-spline technique

We devide the interval $[p, q]$ into N number of subinterval and choose piecewise uniform mesh points denoted by i.e. $y_i = y_0 + ih \ (i = 0,1,\dots,N),$ such that $y_0 = p$ and $y_N = q$ and $h = \frac{q-p}{N}$. Let $L_2[p, q]$ be a vector space of all integrable function on $[p, q]$ and Y be a linear subspace of $L_2[p, q]$. Now we define the septic B-spline basis functions $B_i(y)$ for $i = 0,1,\dots,N-1,N.$

$$B_i(y)=\frac{1}{h^7}\begin{cases}(y-y_{i-4})^7, & y\in[y_{i-4},\ y_{i-3}],\\ (y-y_{i-4})^7-8(y-y_{i-3})^7, & y\in[y_{i-3},\ y_{i-2}],\\ (y-y_{i-4})^7-8(y-y_{i-3})^7+28(y-u_{i-2})^7, & y\in[y_{i-2},\ y_{i-1}],\\ (y-y_{i-4})^7-8(y-y_{i-3})^7+28(y-y_{i-2})^7-56(y-y_{i-1})^7, & y\in[y_{i-1},\ y_i],\\ (y_{i+4}-y)^7-8(y_{i+3}-y)^7+28(y_{i+2}-y)^7-56(y_{i+1}-y)^7, & y\in[y_i,\ y_{i+1}],\\ (y_{i+4}-y)^7-8(y_{i+3}-y)^7+28(y_{i+2}-y)^7, & y\in[y_{i+1},\ y_{i+2}],\\ (y_{i+4}-y)^7-8(y_{i+3}-y)^7, & y\in[y_{i+2},\ y_{i+3}],\\ (y_{i+4}-y)^7, & y\in[y_{i+3},\ y_{i+4}],\\ 0, & \text{otherwise}\end{cases} \tag{4}$$

Let us introduces eight additional mesh points as $y_{-4}<y_{-3}<y_{-2}<y_{-1}<y_0$ and $y_{N+4}>y_{N+3}>y_{N+2}>y_{N+1}>y_N$. From Eq. (4), we easily say that each of the function $B_i(y)$ is six times continuously differentiable in the entire real line. Also, the values of $B_i(y), B_i'(y), B_i''(y), B_i'''(y), B_i^{iv}(y)$ and $B_i^{v}(y)$ at mesh points are given by table 1.

Table 1: Values of $B_i(y), B_i'(y), B_i''(y), B_i'''(y), B_i^{iv}(y)$ and $B_i^{v}(y)$ at nodes

y	y_{i-4}	y_{i-3}	y_{i-2}	y_{i-1}	y_i	y_{i+1}	y_{i+2}	y_{i+3}	y_{i+4}
$B_i(y)$	0	1	120	1191	2416	1191	120	1	0
$hB_i'(y)$	0	7	392	1715	0	-1715	-392	**-7**	0
$h^2B_i''(y)$	0	42	1008	630	-3360	630	1008	42	0
$h^3B_i'''(y)$	0	210	1680	-3990	0	3990	-1680	-210	0
$h^4B_i^{iv}(y)$	0	840	0	-7560	13440	-7560	0	840	0
$h^5B_i^{v}(y)$	0	2520	-10080	12600	0	-12600	10080	-2520	0

Since each $B_i(y)$ is also a piecewise septic polynomial with knots at π, each $B_i(y)\in S_7(\pi)$. Let $\Omega=\{B_{-3},B_{-2},...,B_{N+2},B_{N+3}\}$ and let $B_7(\pi)=span\,\Omega$. The functions Ω are linearly independent on $[p,q]$. Thus $B_7(\pi)$ is $(N+7)$-dimensional. Let $S(y)$ be the B-spline interpolating function at the nodal points and $S(y)\in B_7(\pi)$. Then $S(y)$ can be written as

$$S(y)=\sum_{i=-3}^{N+3} d_iB_i(y) \tag{5}$$

The spline function and their derivative at the mesh points at $y=y_i$ are as follows:

$$S(y_i)=d_{i-3}+120d_{i-2}+1191d_{i-1}+2416d_i+1191d_{i+1}+120d_{i+2}+d_{i+3} \tag{6}$$

$$S'(y_i)=\frac{1}{h}\left(-7d_{i-3}-392d_{i-2}-1715d_{i-1}+1715d_{i+1}+392d_{i+2}+7d_{i+3}\right) \quad (7)$$

$$S''(y_i)=\frac{1}{h^2}\left(42d_{i-3}+1008d_{i-2}+630d_{i-1}-3360d_i+630d_{i+1}+1008d_{i+2}+42d_{i+3}\right) \quad (8)$$

$$S'''(y_i)=\frac{1}{h^3}\left(-210d_{i-3}-1680d_{i-2}+3990d_{i-1}-3990d_{i+1}+1680d_{i+2}+210d_{i+3}\right) \quad (9)$$

$$S^{iv}(y_i)=\frac{1}{h^4}\left(840d_{i-3}-7560d_{i-1}+13440d_i-7560d_{i+1}+840d_{i+3}\right) \quad (10)$$

$$S^{v}(y_i)=\frac{1}{h^5}\left(-2520d_{i-3}+10080d_{i-2}-12600d_{i-1}+12600d_{i+1}-10080d_{i+2}+2520d_{i+3}\right) \quad (11)$$

Discretizing Eq.(1) at the nodal points $y_i\left(i=0,1,\,...,N\right)$, we have

$$-\varepsilon u^{iv}(y_i)+a(y_i)u(y_i)=r(y_i)$$

Using equations (6), and (10) in above equation, we have

$$\begin{aligned}&-\frac{\varepsilon}{h^4}\left(840d_{i-3}-7560d_{i-1}+13440d_i-7560d_{i+1}+840d_{i+3}\right)\\&+a_i\left(d_{i-3}+120d_{i-2}+1191d_{i-1}+2416d_i+1191d_{i+1}+120d_{i+2}+d_{i+3}\right)=r_i\end{aligned} \quad (12)$$

where $a_i=a(y_i)$ and $r_i=r(y_i)$ are the values of $a(y)$ and $r(y)$ at the nodal points $y_i\left(i=0,1,...,N\right)$ and after simplifying Eq. (12). We obtain

$$\begin{aligned}&w_1(y_i)d_{i-3}+w_2(y_i)d_{i-2}+w_3(y_i)d_{i-1}+w_4(y_i)d_i\\&+w_5(y_i)d_{i+1}+w_6(y_i)d_{i+2}+w_7(y_i)d_{i+3}=r_ih^4,\end{aligned} \quad (13)$$

where

$$\begin{aligned}&w_1(y_i)=-840\varepsilon+a_ih^4,\ \ w_2(y_i)=120a_ih^4,\ \ w_3(y_i)=7560\varepsilon+1191a_ih^4,\\&w_4(y_i)=-13440\varepsilon+2416a_ih^4,\quad w_5(y_i)=7560\varepsilon+1191a_ih^4,\\&w_6(y_i)=120a_ih^4,\quad w_7(y_i)=-840\varepsilon+a_ih^4,\qquad i=0,1,\ldots,N.\end{aligned}$$

From Eq. (2), we obtain

$$d_{-3}+120d_{-2}+1191d_{-1}+2416d_0+1191d_1+120d_2+d_3=\alpha_1 \quad (14)$$

$$-7d_{-3}-392d_{-2}-1715d_{-1}+1715d_1+392d_2+7d_3=h\alpha_2 \quad (15)$$

$$-7d_{N-3}-392d_{N-2}-1715d_{N-1}+1715d_{N+1}+392d_{N+2}+7d_{N+3}=h\beta_2 \quad (16)$$

$$d_{N-3}+120d_{N-2}+1191d_{N-1}+2416d_N+1191d_{N+1}+120d_{N+2}+d_{N+3}=\beta_1 \quad (17)$$

Now, two more equations are requrited, Differentiating Eq. (1) with respect to y and after simplifying, we get

$$-\varepsilon u^{v}(y_i)+a(y_i)u'(y_i)+a'(y_i)u(y_i)=r'(y_i). \qquad (18)$$

Putting $y=p$ in above Eq., we have

$$-\varepsilon u^{v}(p)+a(p)u'(p)+a'(p)u(p)=r'(p). \qquad (19)$$

Using Eqs, (6), (7) and (11) in Eq. (19), we obtain

$$\begin{aligned}&\eta_1(p)d_{-3}+\eta_2(p)d_{-2}+\eta_3(p)d_{-1}+\eta_4(p)d_0\\&+\eta_5(p)d_1+\eta_6(p)d_2+\eta_7(p)d_3=h^5r'_p,\end{aligned} \qquad (20)$$

where

$\eta_1(p)=2520\varepsilon-7a(p)h^4+a'(p)h^5, \quad \eta_2(p)=-10080\varepsilon-392a(p)h^4+120a'(p)h^5,$
$\eta_3(p)=12600\varepsilon-1715a(p)h^4+1191a'(p)h^5, \quad \eta_4(p)=2416a'(p)h^5,$
$\eta_5(p)=-12600\varepsilon+1715a(p)h^4+1191a'(p)h^5, \eta_6(p)=10080\varepsilon+392a(p)h^4+120a'(p)h^5,$
$\eta_7(p)=-2520\varepsilon+7a(p)h^4+a'(p)h^5, \quad r'_p=r'(p).$

Similarly, substituting $y=q$ in Eq. (18), we get

$$\begin{aligned}&\lambda_1(q)d_{N-3}+\lambda_2(q)d_{N-2}+\lambda_3(q)d_{N-1}+\lambda_4(q)d_N\\&+\lambda_5(q)d_{N+1}+\lambda_6(q)d_{N+2}+\lambda_7(q)d_{N+3}=h^5r'_q,\end{aligned} \qquad (21)$$

where

$\lambda_1(q)=2520\varepsilon-7a(q)h^4+a'(q)h^5, \quad \lambda_2(q)=-10080\varepsilon-392a(q)h^4+120a'(q)h^5,$
$\lambda_3(q)=12600\varepsilon-1715a(q)h^4+1191a'(q)h^5, \quad \lambda_4(q)=2416a'(q)h^5,$
$\lambda_5(q)=-12600\varepsilon+1715a(q)h^4+1191a'(q)h^5, \lambda_6(q)=10080\varepsilon+392a(q)h^4+120a'(q)h^5,$
$\lambda_7(q)=-2520\varepsilon+7a(q)h^4+a'(q)h^5, \quad r'_q=r'(q).$

Coupling Eqs. (13)-(17) and (20)-(21) in matrix form $Ay=B$, we have $(N+7)$ linear equations with $(N+7)$ unknowns, where A is a non-singular square matrix of order $(N+7)$, y and B are column matrices of order $(N+7)$ i.e. $y=[d_{-3},d_{-2},d_{-1},d_0,d_1,\ldots,d_N,d_{N+1},d_{N+2},d_{N+3}]^T$,
$B=\left[\alpha_1,\alpha_2h, f'_ph^5, f_0h^4, f_1h^4,\ldots, f_Nh^4, f'_qh^5,\beta_2h,\beta_1\right]^T$ and the coefficient matrix A is given by

$$\begin{bmatrix}
1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-7 & -392 & -1715 & 0 & 1715 & 392 & -7 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\eta_1(p) & \eta_2(p) & \eta_3(p) & \eta_4(p) & \eta_5(p) & \eta_6(p) & \eta_7(p) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
w_1(y_0) & w_2(y_0) & w_3(y_0) & w_4(y_0) & w_5(y_0) & w_6(y_0) & w_7(y_0) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & w_1(y_1) & w_2(y_1) & w_3(y_1) & w_4(y_1) & w_5(y_1) & w_6(y_1) & w_7(y_1) & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & w_1(y_2) & w_2(y_2) & w_3(y_2) & w_4(y_2) & w_5(y_2) & w_6(y_2) & w_7(y_2) & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & w_1(y_i) & w_2(y_i) & w_3(y_i) & w_4(y_i) & w_5(t_j) & w_6(y_i) & w_7(y_i) & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & w_1(y_{N-1}) & w_2(y_{N-1}) & w_3(y_{N-1}) & w_4(y_{N-1}) & w_5(y_{N-1}) & w_6(y_{N-1}) & w_7(y_{N-1}) & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & w_1(y_N) & w_2(y_N) & w_3(y_N) & w_4(y_N) & w_5(y_N) & w_6(y_N) & w_7(y_N) \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \lambda_1(q) & \lambda_2(q) & \lambda_3(q) & \lambda_4(q) & \lambda_5(q) & \lambda_6(q) & \lambda_7(q) \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -7 & -392 & -1715 & 0 & 1715 & 392 & 7 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1
\end{bmatrix}.$$

On solving the system of equations $Ay = B$ for unknowns i.e. $d_{-3}, d_{-2}, d_{-1}, d_0, \ldots, d_N, d_{N+1}, d_{N+2}, d_{N+3}$ and putting these values in Eq. (5). After simplifying, we obtain the required numerical solution. All the mathematical calculations are done by MATLAB software.

3. Convergence of Septic B-spline Method

In this section, Septic B-spline technique is deployed to estimate the truncation error over the interval $[p, q]$. Here, we suppose that the function $u(y)$ and their derivatives are continuous in the entire interval.

We compute the following equations by equating the coefficients of $d_i\,(i = -3, \ldots, N+3)$ from Eqs. (6) and (7), we get

$$\begin{aligned}&S'(y_{i-3}) + 120S'(y_{i-2}) + 1191S'(y_{i-1}) + 2416S'(y_i) + 1191S'(y_{i+1}) + 120S'(y_{i+2}) + S'(y_{i+3}) \\ &= \frac{1}{h}\{-7u(y_{i-3}) - 392u(y_{i-2}) - 1715u(y_{i-1}) + 1715u(y_{i+1}) + 392u(y_{i+2}) + 7u(y_{i+3})\}\end{aligned} \tag{22}$$

Similarly from Eqs. (6), (8), (9), (10) and (11), we obtain

$$\begin{aligned}&S''(y_{i-3}) + 120S''(y_{i-2}) + 1191S''(y_{i-1}) + 2416S''(y_i) + 1191S''(y_{i+1}) + 120S''(y_{i+2}) + S''(y_{i+3}) \\ &= \frac{1}{h^2}\{42u(y_{i-3}) + 1008u(y_{i-2}) + 630u(y_{i-1}) - 3360u(y_i) + 630u(y_{i+1}) + 1008u(y_{i+2}) + 42u(y_{i+3})\}\end{aligned} \tag{23}$$

$$\begin{aligned}&S'''(y_{i-3}) + 120S'''(y_{i-2}) + 1191S'''(y_{i-1}) + 2416S'''(y_i) + 1191S'''(y_{i+1}) + 120S'''(y_{i+2}) + S'''(y_{i+3}) \\ &= \frac{1}{h^3}\{-210u(y_{i-3}) - 1680u(y_{i-2}) + 3990u(y_{i-1}) - 3990u(y_{i+1}) + 1680u(y_{i+2}) + 210u(y_{i+3})\}\end{aligned} \tag{24}$$

$$\begin{aligned}&S^{iv}(y_{i-3}) + 120S^{iv}(y_{i-2}) + 1191S^{iv}(y_{i-1}) + 2416S^{iv}(y_i) + 1191S^{iv}(y_{i+1}) + 120S^{iv}(y_{i+2}) + S^{iv}(y_{i+3}) \\ &= \frac{1}{h^4}\{840u(y_{i-3}) - 7560u(y_{i-1}) + 13440u(y_i) - 7560u(y_{i+1}) + 840u(y_{i+3})\}\end{aligned} \tag{25}$$

$$\begin{aligned}&S^{v}(y_{i-3}) + 120S^{v}(y_{i-2}) + 1191S^{v}(y_{i-1}) + 2416S^{v}(y_i) + 1191S^{v}(y_{i+1}) + 120S^{v}(y_{i+2}) + S^{v}(y_{i+3}) \\ &= \frac{1}{h^5}\{-2520u(y_{i-3}) + 10080u(y_{i-2}) - 12600u(y_{i-1}) + 12600u(y_{i+1}) - 10080u(y_{i+2}) + 2520u(y_{i+3})\}\end{aligned} \tag{26}$$

Using the operator notation [6, 9], the equation (22)-(26) can we write as

$$S'(y_i)=\frac{1}{h}\left(\frac{-7E^{-3}-392E^{-2}-1715E^{-1}+1715E+392E^{2}+7E^{3}}{E^{-3}+120E^{-2}+1191E^{-1}+2416I+1191E+120E^{2}+E^{3}}\right)u(y_i) \quad (27)$$

$$S''(y_i)=\frac{1}{h^2}\left(\frac{42E^{-3}+1008E^{-2}+630E^{-1}-3360I+630E+1008E^{2}+42E^{3}}{E^{-3}+120E^{-2}+1191E^{-1}+2416I+1191E+120E^{2}+E^{3}}\right)u(y_i) \quad (28)$$

$$S'''(y_i)=\frac{1}{h^3}\left(\frac{-210E^{-3}-1680E^{-2}+3990E^{-1}-3990E+1680E^{2}+210E^{3}}{E^{-3}+120E^{-2}+1191E^{-1}+2416I+1191E+120E^{2}+E^{3}}\right)u(y_i) \quad (29)$$

$$S^{iv}(y_i)=\frac{1}{h^4}\left(\frac{840E^{-3}-7560E^{-1}+13440I-7560E+840E^{3}}{E^{-3}+120E^{-2}+1191E^{-1}+2416I+1191E+120E^{2}+E^{3}}\right)u(y_i) \quad (30)$$

$$S^{v}(y_i)=\frac{1}{h^5}\left(\frac{-2520E^{-3}+10080E^{-2}-12600E^{-1}+12600E-10080E^{2}+2520E^{3}}{E^{-3}+120E^{-2}+1191E^{-1}+2416I+1191E+120E^{2}+E^{3}}\right)u(y_i) \quad (31)$$

where the operators are defined as $Eu(y_i)=u(y_i+h)$, $Du(y_i)=u'(y_i)$ and $Iu(y_i)=u(y_i)$. Let $E=e^{hD}$ and expand them in powers of hD, we get

$$S'(y_i)=u'(y_i)-\frac{h^8}{151200}u^{ix}(y_i)+\frac{h^{10}}{399168}u^{xi}(y_i)+o[h]^{11}. \quad (32)$$

$$S''(y_i)=u''(y_i)-\frac{h^6}{560}u^{viii}(y_i)+\frac{109h^8}{181440}u^{x}(y_i)-\frac{1039h^{10}}{9979200}u^{xii}(y_i)+o[h]^{11}. \quad (33)$$

$$S'''(y_i)=u'''(y_i)+\frac{h^6}{6048}u^{ix}(y_i)-\frac{h^8}{33600}u^{xi}(y_i)-\frac{41h^{10}}{2177280}u^{xiii}(y_i)+o[h]^{11}. \quad (34)$$

$$S^{iv}(y_i)=u^{iv}(y_i)+\frac{h^4}{720}u^{viii}(y_i)-\frac{h^6}{3024}u^{x}(y_i)+\frac{17h^8}{604800}u^{xii}(y_i)+\frac{h^{10}}{51840}u^{xiv}(y_i)+o[h]^{11}. \quad (35)$$

$$S^{v}(y_i)=u^{v}(y_i)-\frac{h^4}{240}u^{ix}(y_i)+\frac{h^6}{3024}u^{xi}(y_i)-\frac{61h^8}{226800}u^{xiii}(y_i)+\frac{101h^{10}}{1360800}u^{xv}(y_i)+o[h]^{11}, \quad (36)$$

We now define $e(y)=u(y)-S(y)$ and putting the values of Eqs. (32)-(36) in the Taylor series expansion of $e(y_i+th)$ obtaining

$$e(y_i+th)=\frac{t^2(108-7t^2)}{120960}h^8u^{viii}(y_i)+\left(\frac{12t+50t^3-63t^5}{1814400}\right)h^9u^{ix}(y_i) -\frac{t^2(1-5t^2)}{362880}h^{10}u^{x}(y_i)+o[h]^{11}, \quad (37)$$

where $p<t<q$.

Theorem: 1. Let $u(y)$ be the exact solution and $S(u)$ be the numerical treatment of FOSA SPBVPs (1) and BC (2) for smaller values of h which provide the truncation error of $O(h^8)$, and scheme of convergence is $O(h^4)$.

4. Numerical Examples

In this section, a SBSM is portrayed to evaluate the numerical solution of FOSA SPBVPs. It is applied on two examples and numerical results are compared with previous applied methods.

Example: 1. Consider the following boundary value problem:

$$\begin{aligned}-\varepsilon u^{iv}(y)+pu(y)=(y-1)^4 y^8\sin(\varepsilon y)-\varepsilon y^4\Big\{&-16\varepsilon^3(y-1)^3 y^3(3y-2)\cos(\varepsilon y)\\&+96\varepsilon y\left(14-84y+180y^2-165y^3+55y^4\right)\cos(\varepsilon y)\\&+\varepsilon^4(y-1)^4 y^4\sin(\varepsilon y)-24\varepsilon^2(y-1)^2 y^2\left(14-44y+33y^2\right)\sin(\varepsilon y)\\&+24\left(70-504y+1260y^2+320y^3+495y^4\right)\sin(\varepsilon y)\Big\},\qquad y\in[0,\ 1].\end{aligned}$$

with the BCs:

$$u(0)=0,\quad u(1)=0,\quad u'(0)=0,\quad u'(1)=0.$$

Table: 2. MAE and OC of example 1 for various values of ε and h.

ε	$N=20$	$N=40$	$N=80$	$N=160$
$1/20$	5.2815E-08	3.8482E-09	2.4916E-10	1.5705E-11
OC	3.7787	3.9491	3.9877	3.9724
$1/40$	2.7322E-08	1.9911E-09	1.2892E-10	8.1281E-12
OC	3.7784	3.9490	3.9874	3.9823
$1/80$	1.4731E-08	1.0737E-09	6.9520E-11	4.3845E-12
OC	3.7782	3.9490	3.9870	4.0460
$1/160$	8.8218E-09	6.4640E-10	4.1921E-11	2.6435E-12
OC	3.7706	3.9467	3.9871	3.9959

Table: 3. Comparison of MAE of example 1 for various values of ε and h.

ε	$h=1/64$			$h=1/128$		
	Present Method	Akram and Naheed [1]	Akram and Amin [2]	Present Method	Akram and Naheed [1]	Akram and Amin [2]
$1/16$	7.50E-10	2.61E-09	2.85E-08	4.75E-11	6.72E-11	6.68E-09
$1/32$	3.86E-10	1.34E-09	1.43E-08	2.44E-11	3.45E-11	3.36E-09
$1/64$	2.04E-10	7.13E-10	7.25E-09	1.29E-11	1.83E-11	1.69E-09
$1/128$	1.17E-10	4.09E-10	3.72E-09	7.42E-12	1.05E-11	8.62E-10

Exact solution of the example 1 is

$$u(y)=(1-y)^4 y^8 \sin(\varepsilon y).$$

Numerical results of example 1 depected in tables 2, 3 and 4. Table 2 displays the maximum absolute error (MAE) and order of convergence (OC) for different values of ε and h. Table 3 shows comparison with existing methods and table 4 presents point wise numerical and exact solution for $\varepsilon=10^{-2}$ and $N=100$.

Table: 4. Pointwise solution of example 1 for $\varepsilon=10^{-2}$ and $N=100$.

y	Numerical solution	Exact solution
0.00	0.0000E-00	00000000
0.09	1.2042E-12	2.6567E-12
0.20	2.0910E-09	2.0972E-09
0.40	3.3972E-07	3.3974E-07
0.50	1.2207E-06	1.2207E-06
0.60	2.5798E-06	2.5799E-06
0.70	3.2686E-06	3.2686E-06
0.80	2.1474E-06	2.1475E-06
0.90	3.8741E-07	3.8742E-07
0.99	9.1202E-11	9.1350E-11
1.00	0.0000E-00	0.0000E-00

Table: 5. MAE and OC of example 2 for various values of ε and h.

ε	$N=16$	$N=32$	$N=64$	$N=128$
1/16	4.5559E-04	2.8450E-05	1.7785E-06	1.1116E-07
OC	4.0012	3.9997	3.9999	4.0060
1/32	1.0414E-04	6.5464E-06	4.0900E-07	2.5561E-08
OC	3.9918	4.0005	4.0001	3.9968
1/64	4.9514E-05	3.1226E-06	1.9504E-07	1.2187E-08
OC	3.9870	4.0009	4.0003	3.9924
1/128	2.2966E-05	1.4366E-06	8.9900E-08	5.6183E-09
OC	3.9988	3.9982	4.0001	4.0001

Example: 2. Consider the following boundary value problem:

$$-\varepsilon u^{iv}(y)+p(y)u(y)=\varepsilon\left((y-1)^4 y^4-24\varepsilon\left(5-60y+210y^2-280y^3+126y^4\right)\right),\quad y\in[-1,1]$$

with the BCs:

$$u(-1) = -16\varepsilon, \quad u(1) = 0, \quad u''(-1) = -688\varepsilon, \quad u''(1) = 0.$$

Exact solution of the example 2 is $u(y) = \varepsilon y^5 (1-y)^4$.

Numerical results of example 2 are depicted in table 5, 6 and 7. Table 5 displays the MAE and OC for different values of ε and h. Table 6 shows comparison with existing methods and table 7 presents point wise numerical and exact solution for $\varepsilon = 10^{-2}$ and $N = 100$.

Table: 6. Comparison of MAE of example 2 for various values of ε and h.

ε	$h = 1/40$			$h = 1/80$		
	Present Method	Akram and Naheed [1]	Akram and Amin [2]	Present Method	Akram and Naheed [1]	Akram and Amin [2]
$1/16$	7.28E-07	2.95E-06	8.10E-03	4.55E-08	8.05E-08	2.00E-03
$1/32$	1.68E-07	9.20E-07	2.00E-03	1.05E-08	2.25E-08	4.95E-04
$1/64$	7.99E-08	6.47E-07	1.00E-03	4.99E-09	1.42E-08	2.59E-04
$1/128$	3.68E-08	3.39E-07	4.93E-04	2.30E-09	7.20E-09	1.24E-04

Table: 7. Pointwise solution of example 2 for $\varepsilon = 10^{-2}$ and $N = 100$.

y	Numerical solution	Exact solution
-1.0	-1.6000E-01	-1.6000E-01
-0.9	-7.6953E-02	-7.6953E-02
-0.5	-1.5822E-03	-1.5820E-03
-0.1	-2.1286E-07	-1.4641E-07
0.1	1.2180E-07	6.5610E-08
0.5	1.9726E-05	1.9531E-05
0.9	6.5103E-07	5.9049E-07
1.0	0.0000E-00	0.0000E-00

5. Conclusion

In this study, we have discussed a SBSM to estimate the numerical solution of SAFO SPBVPs. This methed is inmplimented on two numerical examples and threir results are shown in tables 2, 3, 4, 5, 6 and 7 through these results we can easily say that proposed method gives the better numerical results as compared existing methods [1, 2] at the same values of ε and h. Moreover, SBSM is computationally capble and the algorithm can be effortlessly executed on computer.

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