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Fixed Point Theorems for Expansive Mapping in A-Metric Space

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ABSTRACT

In this paper, we prove some fixed point theorems under different expansive type conditionsin the setting of a A-metric space. Our results generalize and extend various results in the existing literature.

Keywords: A-metric space, expansive mapping, fixed point.

1. INTRODUCTION

The study of expansive mappings is very interesting research area of fixed point theory. In 1984, Wang *et al.*[26] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko[7] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. In 1989, Bakhtin[2] introduced the concept of a b-metric space as a generalization of metric spaces, in which many researchers treated the fixed point theory. In 1993, Czerwik[4-5] extended many results related to the b-metric spaces. In 1994, Matthews[14] introduced the concept of partial metric space in which the self-distance of any point of space may not be zero. Gähler[11] claimed that 2-metric space is a generalization of an ordinary metric space. He mentioned in[11] that $d(x, y, z)$ geometrically represents the area of a triangle formed by the points $x, y, z \in X$as its vertices. On the other hand, Ha *et al.*[12] and Sharma[23] found some mathematical flaws in these claims. It was demonstrated in[23] that $d(x, y, z)$ does not always represent the area of a triangle formed by the points $x, y, z \in X$. Dhage[8] suggested an improvement in the basic structure of 2-metric space. In 1984, Dhage in his Ph.D. thesis[8] identified condition second as a weakness in Gähler's theory of a 2-metric space. To overcome these problems, he then introduced the concept of a *D*-metric space. Dhage[9] then studied topological properties of *D*-metric space in a series of papers. Naidu *et al.*[18] proved that the concepts of convergent sequences and sequential continuity are not well defined in *D*-metric spaces. Naidu *et al.*[19] pointed out some drawbacks in the idea of open balls in *D*-metric space. In 2003, Mustafa and Sims[17] identified condition third as a weakness in Dhage's theory of *D*-metric space. The tetrahedral inequality in *D*-metric has been replaced with the prototypical rectangular inequality adopted by Mustafa and Sims[16] in 2006 and introduced the notion of *G*-metric space and suggested an important generalization of metric space. Sedghi *et al*.[20] have introduced D^*-metric spaces which is a probable modification of the definition of *D*-metric spaces introduced by Dhage[8] and proved some basic properties in D^*-metric spaces, (see[22]). Every *G*-metric space is a D^*-metric space. The converse, however, is false in general; a D^*-metric space is not necessarily a *G*-metric space. Sedghi *et al*.[21] identified condition third of the *G*-metric space as a peculiar limitation but classified the symmetry condition as a common weakness of both *G*- and D^*-metric spaces. To overcome these difficulties, Sedghi *et al*.[21] introduced a new generalized metricspace called an *S*-metric space. The S-metric space is a space with three dimensions. Sedghi *et al*.[21] asserted that every G-metric is an S-metric, see[21], Remarks 1.3 and [21], Remarks 2.2. The Example 2.1 and Example 2.2 of Dung *et al*.[10] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct. Souayah *et al*.[25] have introduced S_b-metric space and established some fixed point theorems. Very recently, Abbas *et al*.[15] introduced the notion of A-metric space, which generalization of the *S*-metric space.

In this paper, we prove some fixed point theorems under expansive type conditions in the setting of a *A*-metric space. Our results generalize and extend various results in the existing literature.

2. PRELIMINARIES

In 2015, Abbas *et al*.[15] introduced the notion of*A*-metric space.

Definition 2.1(see[15])Let Xbe a nonempty set. A mapping $A: X^n \to [0,+\infty)$ is called an A-metric on X if and only if for all $x_i, a \in X, i = 1,2,3,..n$: the following conditions hold:

(A1). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$,

(A2). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$,

(A3). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a)$
$+A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a)$
$+A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \cdots$
$+A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a)$
$+A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]$.

The pair (X, A) is called an A-metric space.

The following is the intuitive geometric example for A-metric spaces.

Example 2.2(see[15])Let $X = [1,+\infty)$. Define $A: X^n \to [0,+\infty)$ by

$$A(x_1, x_2, x_3, \dots., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|$$

for all $x_i \in X, i = 1,2, \dots n$.

Example2.3 (see[15])Let $= \mathbb{R}$. Define $A: X^n \to [0,+\infty)$ by

$$A(x_1, x_2, x_3, \dots., x_{n-1}, x_n) = \left|\sum_{i=n}^{2} x_i - (n-1)x_1\right|$$
$$+ \left|\sum_{i=n}^{3} x_i - (n-2)x_2\right| + \cdots$$
$$+ \left|\sum_{i=n}^{n-3} x_i - 3x_{n-3}\right|$$
$$+ \left|\sum_{i=n}^{n-2} x_i - 2x_{n-2}\right|$$
$$+ |x_n - x_{n-1}|$$

for all $x_i \in X, i = 1,2, \dots n$.

Lemma 2.4 (see[15]) Let (X, A) be an A-metric space. Then for all $x, y \in X$,
$A_b(x, x, x, x, \dots, (x)_{n-1}, y) = A_b(y, y, y, y \dots., (y)_{n-1}, x)$

Lemma 2.5 (see[15]) Let (X, A) be an A-metric space. Then for all $x, y, z \in X$,
$A_b(x, x, x, x, \dots, (x)_{n-1}, z) \le (n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y)$
$+A_b(z, z, z, z, \dots, (z)_{n-1}, y)$
and
$A_b(x, x, x, x, \dots, (x)_{n-1}, z) \le (n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y)$
$+A_b(y, y, y, y, \dots, (y)_{n-1}, z)$

Lemma 2.6(see[15])Let (X, A) be an A-metric space. Then $(X \times X, D_A)$ is an A-metric space on $X \times X$, where D_A is given by for all $x_i, y_j \in X, i, j = 1,2, \dots, n$:
$D_A\big((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)\big)$
$= A(x_1, x_2, x_3, \dots., x_n) + A(y_1, y_2, y_3, \dots., y_n)$.

Definition 2.7(see[15])Let (X, A) be an A-metric space. Then

1. A sequence $\{x_k\}$ is called convergent to x in (X, A) if
$\lim_{k\to+\infty} A(x_k, x_k, x_k, x_k, .. (x_k)_{n-1}, x) = 0.$
That is, for each $\epsilon \ge 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \ge n_0$, we have
$A(x_k, x_k, x_k, x_k, .. (x_k)_{n-1}, x) \le \epsilon$
and we write $\lim_{k\to+\infty} x_k = x$.
2. A sequence $\{x_k\}$ is called Cauchy in (X, A) if
$\lim_{k,m\to+\infty} A(x_k, x_k, x_k, x_k, .. (x_k)_{n-1}, x_m) = 0.$
That is, for each $\epsilon \ge 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \ge n_0$, we have
$A(x_k, x_k, x_k, x_k, .. (x_k)_{n-1}, x_m) \le \epsilon.$

3. (X, A) is said to be complete if every Cauchy sequence in (X, A) is a convergent.

Lemma 2.8(see[15]) Let (X, A) be an A-metric space. If the sequence $\{x_k\}$in Xconverges to x, then $x$is unique.

Lemma 2.9(see[15])Every convergent sequence in A-metric space (X, A, s) is a Cauchy sequence.

Definition 2.10 Let (X, A_b)be an A_b-metric spacewith $s \geq 1$. A map $f: X \to X$ is said to be expansive mapping if there exits $\lambda > 1$ such that

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \geq \lambda A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$.

3.1 MAİN RESULT

We begin with a simple but a useful lemma.

Lemma 3.1 Let (X, A) be an A-metric space and $\{x_k\}$be a sequence in (X, A) such that
$A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_{k+1}) \leq \lambda A(x_{k-1}, x_{k-1}, x_{k-1}, .. (x_{k-1})_{n-1}, x_k)$ (3.1)
where $\lambda \in [0, 1)$and $k = 1, 2, \dots$. Then $\{x_k\}$is a Cauchy sequence in (X, A).

Proof For $k = 1, 2, ..$, we get by induction
$A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_{k+1}) \leq \lambda A(x_{k-1}, x_{k-1}, x_{k-1}, .. (x_{k-1})_{n-1}, x_k)$
$\leq \lambda^2 A(x_{k-2}, x_{k-2}, x_{k-2}, .. (x_{k-2})_{n-1}, x_{k-1})$
:
:
$\leq \lambda^k A(x_0, x_0, x_0, x_0, .. (x_0)_{n-1}, x_1)$ (3.2)
Let $m > k$. It follows that
$A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_m)$
$\leq [(n-1)A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_{k+1})$
$+A(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1})]$
$\leq (n-1)A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_{k+1})$
$+A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_m)$
$\leq (n-1)A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_{k+1})$
$+[(n-1)A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2})$
$+A(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+2})]$
$\leq (n-1)A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_{k+1})$
$+(n-1)A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2})$
$+A(x_{k+2}, x_{k+2}, x_{k+2}, \dots, (x_{k+2})_{n-1}, x_m)]$
$\leq (n-1)A(x_k, x_k, x_k, .. (x_k)_{n-1}, x_{k+1})$
$+(n-1)A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2})$
$+[(n-1)A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+3})$

$+A(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+3})]$
$\leq (n-1)A(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1})$
$+(n-1)A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2})$
$+(n-1)A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+3})$
$+(n-1)A(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+4}) + \cdots$
$+(n-1)A(x_{m-2}, x_{m-2}, x_{m-2}, \dots, (x_{m-2})_{n-1}, x_{m-1})$
$+A(x_{m-1}, x_{m-1}, x_{m-1}, \dots, (x_{m-1})_{n-1}, x_m)$
$\leq (n-1)[\lambda^k + \lambda^{k+1} + \lambda^{k+2} + \lambda^{k+3} + \cdots + \lambda^{m-2}]$
$\times A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1)$
$+\lambda^{m-1} \times A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1)$
$= (n-1)\lambda^k[1 + \lambda + \lambda^2 + \lambda^3 + \cdots + \lambda^{m-k-2}]$
$\times A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1)$
$+\lambda^{m-k-1} \times A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1)$
$\leq (n-1)\lambda^k[1 + \lambda + \lambda^2 + \lambda^3 + \cdots]$
$\times A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1)$
$\leq (n-1)\frac{\lambda^k}{1-\lambda} A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1)$ (3)

Since $\lambda < 1$. Assume that $A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) > 0$. By taking limit as $k, m \to +\infty$ in above inequality we get

$$\lim_{k,m \to +\infty} A(x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) = 0.$$

Therefore, $\{x_k\}$ is a Cauchy sequence in X. Also, if $A(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) = 0$, then $A(x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) = 0$ for all $m > k$ and hence $\{x_k\}$ is a Cauchy sequence in X. Now, our first main results as follows.

Theorem 3.2 Let (X, A) be a complete A-metric space. Assume that the mapping $T: X \to X$ is surjection and satisfies

$$A(Tx^1, Tx^2, Tx^3, \dots .Tx^{n-1}, Tx^n) \geq \lambda A(x^1, x^2, x^3, \dots .x^{n-1}, x^n) \quad (3.3)$$

$\forall x^1, x^2, x^3, \dots .x^{n-1}, x^n \in X$,where$\lambda > 1$. Then T has a fixed point.

Proof Let $x_0 \in X$, since T is surjection on X, then there exists $x_1 \in X$ such that$x_0 = Tx_1$. By continuing this process, we get

$$x_k = Tx_{k+1}, \ \forall k \in \mathbb{N} \cup \{0\}. \quad (3.4)$$

If $A(x_{m-1}, x_{m-1}, x_{m-1}, \dots, x_{m-1}, x_m) = 0$ for some m, then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T. Without loss of generality, we can suppose that$A(x_{k-1}, x_{k-1}, x_{k-1}, \dots ., x_{k-1}x_n) > 0$, that is, $x_k \neq x_{k-1}$ for every k. From (3.3), we have

$A(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_k) = A(Tx_k, Tx_k, Tx_k, \dots ., Tx_{k+1})$
$\geq \lambda A(x_k, x_k, x_k, \dots ., x_{k+1})$

and so

$$A(x_k, x_k, x_k, \dots ., x_{k+1}) \leq \frac{1}{\lambda} A(x_{k-1}, x_{k-1}, x_{k-1}, \dots ., x_k)$$

$= hA(x_{k-1}, x_{k-1}, x_{k-1}, \ldots., x_k)$ (3.5)

for all $k \in \mathbb{N}$, where $h = \frac{1}{\lambda} < 1$. By Lemma 3.1,$\{x_k\}$ is a Cauchy sequence in X. Since X is a complete A-metric space, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $k \to +\infty$. Now since T is surjective map. So there exists a point p in X such that $x^\star = Tp$. From (3.3), we have

$A(x_k, x_k, x_k, \ldots., x_k, x^\star) = A(Tx_{k+1}, Tx_{k+1}, Tx_{k+1}, \ldots, Tx_{k+1}, Tp)$

$\geq \lambda A(x_{k+1}, x_{k+1}, x_{k+1}, \ldots., p)$

Taking limit as $k \to +\infty$ in the above inequality, we get

$0 = \lim_{k\to+\infty} A(x_k, x_k, x_k, \ldots., x_k, x^\star) \geq \lambda \lim_{n\to\infty} A(x_{k+1}, x_{k+1}, x_{k+1}, \ldots., x_{k+1}, p)$

which implies that

$\lim_{n\to+\infty} A(x_{k+1}, x_{k+1}, x_{k+1}, \ldots., x_{k+1}, p) = 0.$ (3.6)

Thus $x_{n+1} \to p$ as $k \to +\infty$. By Lemma 2.8, we get $x^\star = p$. Hence $x^\star$is a fixed point of T. Finally, assume $x^\star = y^\star$ is also another fixed point of T. From (3.3), we get

$A(x^\star, x^\star, x^\star, x^\star, \ldots., (x^\star)_{n-1}, y^\star) = A(Tx^\star, Tx^\star, Tx^\star, Tx^\star, \ldots., (Tx^\star)_{n-1}, y^\star)$

$\geq \lambda A(x^\star, x^\star, x^\star, x^\star, \ldots., (x^\star)_{n-1}, y^\star)$

This is true only when $A(x^\star, x^\star, x^\star, x^\star, \ldots., (x^\star)_{n-1}, y^\star) = 0$.So $x^\star = y^\star$. Hence T has a unique fixed point in X.

Corollary 3.3 Let (X, A) be a complete A-metric space and $T: X \to X$ be a surjection. Suppose that there exist a positive integer k and a real number $\lambda > 1$ such that

$A\left(T^k(x^1), T^k(x^2), \ldots, T^k(x^{n-1}), T^k(x^n)\right) \geq \lambda A(x^1, x^2, \ldots. x^{n-1}, x^n)$ (3.7)

$\forall\, x^1, x^2, x^3, \ldots. x^{n-1}, x^n \in X$. Then T has a fixed point.

Proof From Theorem 3.2, T^k has a fixed point $x^\star$. But $T^k(Tx^\star) = T\left(T^k x^\star\right) = Tx^\star$, So $Tx^\star$ is also a fixed point of T^k. Hence $Tx^\star = x^\star, x^\star$ is a fixed point of T. Since the fixed point of T is also fixed point of T^k , the fixed point of T is unique.

Theorem 3.4 Let (X, A) be a complete A-metric space and T: X → X be a surjection such that

$A(Tx^1, Tx^2, Tx^3, \ldots. Tx^{n-1}, Tx^n) \geq \lambda_1 A(x^1, x^2, x^3, \ldots. x^{n-1}, x^n)$

$+\lambda_2 A(x^1, x^1, x^1, \ldots. (x^1)_{n-1}, Tx^1) + \cdots \ldots$

$+\lambda_{n+1} A(x^n, x^n, x^n, \ldots., (x^n)_{n-1}, Tx^n)$ (3.8)

$\forall\, x^1, x^2, x^3, \ldots. x^{n-1}, x^n \in X$, where $\sum_{i=1}^{n+1} \lambda_i > 1, \lambda_i \geq 0$. Then T has a fixed point.

Proof Let $x_0 \in X$, since T is surjection on X, then there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we get

$x_k = Tx_{k+1},\ \forall\, k \in \mathbb{N} \cup \{0\}.$ (3.9)

If $A(x_{m-1}, x_{m-1}, x_{m-1}, \ldots, x_{m-1}, x_m) = 0$ for some m, then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T. Without loss of generality, we can suppose that $A(x_{k-1}, x_{k-1}, x_{k-1}, \ldots., x_{k-1} x_n) > 0$, that is, $x_k \neq x_{k-1}$ for every k. From (3.8), we have

$A(x_{k-1}, x_{k-1}, x_{k-1}, \ldots, (x_{k-1})_{n-1}, x_k) = A(Tx_k, Tx_k, Tx_k, \ldots., (Tx_k)_{n-1}, Tx_{k+1})$

$\geq \lambda_1 A(x_k, x_k, x_k, \ldots., (x_k)_{n-1}, x_{k+1})$

$+\lambda_2 A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, Tx_k)$
$+\lambda_3 A(x_k, x_k, x_k, \dots. x_k, (x_k)_{n-1}, Tx_k) + \cdots \dots \dots$
$+\lambda_{n+1} A(x_{k+1}, x_{k+1}, x_{k+1}, \dots., (x_{k+1})_{n-1},, Tx_{k+1})$
$= \lambda_1 A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1})$
$+\lambda_2 A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k-1})$
$+\lambda_3 A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k-1}) + \cdots \dots \dots$
$+\lambda_{n+1} A(x_{k+1}, x_{k+1}, x_{k+1}, \dots., (x_{k+1})_{n-1}, x_k)$
$= \lambda_1 A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1})$
$+\lambda_2 A(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k)$
$+\lambda_3 A(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k) + \cdots \dots \dots$
$+\lambda_{n+1} A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1})$

Hence, we have

$$\left(1 - \sum_{i=2}^{n} \lambda_i\right) A(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k) \geq (\lambda_1 + \lambda_{n+1}) A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1})$$

If $\lambda_1 + \lambda_{n+1} = 0$, then $\sum_{i=2}^{n} \lambda_i > 0$. The above inequality implies that a negative number is greater then or equal to zero. This is impossible. So, $\lambda_1 + \lambda_{n+1} \neq 0$ and $(1 - \sum_{i=2}^{n} \lambda_i) > 0$. Therefore,

$$A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1}) \leq \lambda A(x_{k-1}, x_{k-1}, x_{k-1}, \dots., x_k) \quad (3.10)$$

where $\lambda = \frac{(1 - \sum_{i=2}^{n} \lambda_i)}{\lambda_1 + \lambda_{n+1}} < 1$ for all k $\in \mathbb{N} \cup \{0\}$. Repeating (3.10) k-times, we get

$$A(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1}) \leq \lambda^k A(x_0, x_0, x_0, \dots., x_1) \quad (3.11)$$

By Lemma 3.1,$\{x_k\}$ is a Cauchy sequence in X. Since X is a complete A-metric space, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $k \to +\infty$. Now since T is surjective map. So there exists a point p in X such that $x^\star = Tp$. From (3.8), we have

$A(x_k, x_k, x_k, \dots., x_k, x^\star) = A(Tx_{k+1}, Tx_{k+1}, Tx_{k+1}, \dots, Tx_{k+1}, Tp)$
$\geq \lambda_1 A(x_{k+1}, x_{k+1}, x_{k+1}, \dots., p)$
$+\lambda_2 A(x_{k+1}, x_{k+1}, \dots. (x_{k+1})_{n-1}, Tx_{k+1}) + \cdots \dots.$
$+\lambda_{n+1} A(p, p, p, \dots., (p)_{n-1}, Tp)$

Taking limit as $k \to +\infty$ in the above inequality, we get

$$0 = \lim_{k \to +\infty} A(x_k, x_k, x_k, \dots., x_k, x^\star) \geq \lambda_1 \lim_{n \to \infty} A(x_{k+1}, x_{k+1}, x_{k+1}, \dots., x_{k+1}, p)$$
$$+ \sum_{i=2}^{n} \lambda_i \lim_{n \to \infty} A(x_{k+1}, x_{k+1}, \dots. (x_{k+1})_{n-1}, Tx_{k+1})$$
$$+\lambda_{n+1} A(p, p, p, \dots., (p)_{n-1}, Tp)$$
$$= \lambda_1 \lim_{n \to \infty} A(x_{k+1}, x_{k+1}, x_{k+1}, \dots., x_{k+1}, p)$$
$$+ \sum_{i=2}^{n} \lambda_i \lim_{n \to \infty} A(x_{k+1}, x_{k+1}, \dots. (x_{k+1})_{n-1}, Tx_{k+1})$$

$+\lambda_{n+1}A(p,p,p,\ldots.,(p)_{n-1},Tp)$
which implies that
$0 \geq \lambda_1 A(x^\star,x^\star,x^\star,\ldots.,x^\star,p) + \lambda_{n+1}A(p,p,p,\ldots.,(p)_{n-1},x^\star)$.
By using Lemma 2.4, we have
$0 \geq (\lambda_1 + \lambda_{n+1})A(x^\star,x^\star,x^\star,\ldots.,x^\star,p)$
Hence $p = x^\star$.This gives that$x^\star$ is a fixed point of T. This completes the proof.
Finally, assume $x^\star = y^\star$ is also another fixed point of T. From (3.8), we get
$A(x^\star,x^\star,x^\star,x^\star,\ldots.,(x^\star)_{n-1},y^\star) = A(Tx^\star,Tx^\star,Tx^\star,Tx^\star,\ldots.,(Tx^\star)_{n-1},Ty^\star)$
$\geq \lambda_1 A(x^\star,x^\star,x^\star,x^\star,\ldots.,(x^\star)_{n-1},y^\star)$
$+\sum_{i=2}^{n}\lambda_i\, A(x^\star,x^\star,x^\star,x^\star,\ldots.,(x^\star)_{n-1},Tx^\star)$
$+\lambda_{n+1}A(y^\star,y^\star,y^\star,\ldots.,(y^\star)_{n-1},Ty^\star)$
$= \lambda_1 A(x^\star,x^\star,x^\star,x^\star,\ldots.,(x^\star)_{n-1},y^\star)$
This is true only when $A(x^\star,x^\star,x^\star,x^\star,\ldots.,(x^\star)_{n-1},y^\star) = 0$. So $x^\star = y^\star$. Hence T has a unique fixed point in X.

Conflict of Interest

No conflict of ınterest was declared by the authors.

Author's Contributions

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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