Dass and Gupta Rational Type Contraction in Controlled Metric Spaces

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Abstract

The aim of this paper is to establish a fixed point theorem for rational type contraction in a complete controlled metric space. Our results extend/generalize many pre-existing results in literature. We also provide example which show the usefulness of these results.

Keywords: Fixed point theory; Rational type contraction; Controlled metric space.

MSC: 47H10; 54H25

1. Introduction and Preliminaries

Dass and Gupta [26] established first fixed point theorem for rational contractive type conditions in metric space.

Theorem 1.1 (see [26]). Let (X, d) be a complete metric space, and let $\mathcal{T}: X \to X$ be a self-mapping. If there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ such that

$$d(\mathcal{T}x, \mathcal{T}y) \leq \alpha d(x, y) + \beta \frac{[1 + d(x, \mathcal{T}x)]d(y, \mathcal{T}y)}{1 + d(x, y)} \tag{1.1}$$

for all $x, y \in X$, then $\mathcal{T}$ has a unique fixed point $x^* \in X$.

Nazam *et al.* [27] proved a real generalization of Dass-Gupta fixed point theorem in the frame work of dualistic partial metric spaces.

Czerwik [1] reintroduced a new class of generalized metric spaces, called as b-metric spaces, as generalizations of metric spaces.

Definition 1 ([1]). Let X be a nonempty set and $s \geq 1$. A function $d_b: X \times X \longrightarrow [0, \infty)$ is said to be a b -metric if for all $x, y, \omega \in X$,

(b1). $d_b(x, y) = 0$ iff $x = y$
(b2). $d_b(x, y) = d_b(y, x)$ for all $x, y \in X$
(b3). $d_b(x, \omega) \leq s[d_b(x, y) + d_b(y, \omega)]$

The pair (X, d_b) is then called a b-metric space. Subsequently, many fixed point results on such spaces were given (see [2–7]).

Kamran et al. [8] initiated the concept of extended b-metric spaces.

Definition 2. Let X be a nonempty set and $p: X \times X \longrightarrow [1, \infty)$ be a function. A function $d_e: X \times X \longrightarrow [0, \infty)$ is called an extended b -metric if for all $x, y, \omega \in X$,

(e1). $d_e(x, y) = 0$ iff $x = y$
(e2). $d_e(x, y) = d_e(y, x)$ for all $x, y \in X$
(e3). $d_e(x, \omega) \leq p(x, \omega)[d_e(x, y) + d_e(y, \omega)]$

The pair(X, d_e) is called an extended b-metric space.

Very recently, a new kind of a generalized b-metric space was introduced by Mlaiki et al. [9].

Definition 3 ([9]). Let X be a nonempty set and $p: X \times X \longrightarrow [1, \infty)$ be a function. A function $d_c: X \times X \longrightarrow [0, \infty)$ is called a controlled metric if for all $x, y, \omega \in X$,

(c1). $d_c(x, y) = 0$ iff $x = y$
(c2). $d_c(x, y) = d_c(y, x)$ for all $x, y \in X$
(c3). $d_c(x, \omega) \leq p(x, \omega)[d_c(x, y) + d_c(y, \omega)]$

The pair (X, d_c) is called a controlled metric space (see also [10]).

The Cauchy and convergent sequences in controlled metric type spaces are defined in this way

Definition 4 ([9]). Let (X, d_c) be a controlled metric space and $\{x_n\}_{n\geq 0}$be a sequence in D. Then,

1. The sequence $\{x_n\}$ converges to some x in X; if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_c(x_n, x) < \varepsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.
2. The sequence $\{x_n\}$ is Cauchy; if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_c(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.
3. The controlled metric space (X, d_c) is called complete if every Cauchy sequence is convergent.

Definition 5 ([9]). Let (X, d_c) be a controlled metric space. Let $x \in X$ and $\varepsilon > 0$.

1. The open ball $B(x, \varepsilon)$ is
$$B(x, \varepsilon) = \{y \in X: d_c(y, x) < \varepsilon\}.$$
2. The mapping $F: X \longrightarrow X$ is said to be continuous at $x \in X$; if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $F\big(B(x, \varepsilon)\big) \subseteq B(Fx, \varepsilon)$.

The main purpose of this paper is to present some fixed point theorems for mappings involving rational expressions in the context of complete controlled metric spaces. Our result extends and generalizes some well-known results in the literature. We also provide examples to show significance of the obtained results involving rational type contractive conditions.

2 Results on Rational Type Contractions

Theorem 2.1. Let (X, d_c) be a complete controlled metric space. Let $F: X \longrightarrow X$ be so that there are $\gamma_i \in (0,1), \forall\, i \in \{1,2\}$ with $\lambda = \frac{\gamma_2}{1-\gamma_1} < 1$,

$$d_c(Fx, Fy) \leq \gamma_1 \frac{d_c(y,Fy)[1+d_c(x,Fx)]}{1+d_c(x,y)} + \gamma_2 d_c(x, y) \tag{1}$$

for all $x, y \in X$. For $x_0 \in X$, take $x_n = F^n x_0$. Assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{p(x_{i+1},x_{i+2})p(x_{i+1},x_m)}{p(x_i,x_{i+1})} < \lambda^{-1} \tag{2}$$

Suppose that $\lim_{n\to\infty} p(x_n, x)$ and $\lim_{n\to\infty} p(x, x_n)$ exist, are finite, and $\gamma_1 \lim_{n\to\infty} p(x, x_n) < 1$ for every $x \in X$, then F possesses a unique fixed point.

Proof. Let $x_0 \in X$ be initial point. The considered sequence $\{x_n\}$ verifies $x_{n+1} = Fx_n$ for all $n \in \mathbb{N}$. Obviously, if there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then $Fx_{n_0} = x_{n_0}$, and the proof is finished. Thus, we suppose that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$. Thus, by (1), we have

$$\begin{aligned} d_c(x_n, x_{n+1}) &= d_c(Fx_{n-1}, Fx_n) \\ &\leq \gamma_1 \frac{d_c(x_n,Fx_n)[1+d_c(x_{n-1},Fx_{n-1})]}{1+d_c(x_{n-1},x_n)} + \gamma_2 d_c(x_{n-1}, x_n) \\ &= \gamma_1 \frac{d_c(x_n,x_{n+1})[1+d_c(x_{n-1},x_n)]}{1+d_c(x_{n-1},x_n)} + \gamma_2 d_c(x_{n-1}, x_n) \\ &= \gamma_1 d_c(x_n, x_{n+1}) + \gamma_2 d_c(x_{n-1}, x_n) \end{aligned}$$

The last inequality gives

$$d_c(x_n, x_{n+1}) \leq \frac{\gamma_2}{1-\gamma_1} d_c(x_{n-1}, x_n) \tag{3}$$

Thus, we have

$$d_c(x_n, x_{n+1}) \leq \lambda d_c(x_{n-1}, x_n) \leq \lambda^2 d_c(x_{n-2}, x_{n-1}) \leq \cdots \leq \lambda^n d_c(x_0, x_1) \tag{4}$$

For all $n, m \in \mathbb{N}$ and $n < m$, we have

$$d_c(x_n, x_m) \leq p(x_n, x_{n+1}) d_c(x_n, x_{n+1}) + p(x_{n+1}, x_m) d_c(x_{n+1}, x_m)$$

$$\leq p(x_n, x_{n+1})d_c(x_n, x_{n+1}) + p(x_{n+1}, x_m)p(x_{n+1}, x_{n+2})d_c(x_{n+1}, x_{n+2})$$

$$+p(x_{n+1}, x_m)p(x_{n+2}, x_m)d_c(x_{n+2}, x_m)$$

$$\leq p(x_n, x_{n+1})d_c(x_n, x_{n+1}) + p(x_{n+1}, x_m)p(x_{n+1}, x_{n+2})d_c(x_{n+1}, x_{n+2})$$

$$+p(x_{n+1}, x_m)p(x_{n+2}, x_m)p(x_{n+2}, x_{n+3})d_c(x_{n+2}, x_{n+3})$$

$$+p(x_{n+1}, x_m)p(x_{n+2}, x_m)p(x_{n+3}, x_m)d_c(x_{n+3}, x_m)$$

$$\leq p(x_n, x_{n+1})d_c(x_n, x_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} p(x_j, x_m)\right) p(x_i, x_{i+1})d_c(x_i, x_{i+1})$$

$$+ \prod_{i=n+1}^{m-1} p(x_j, x_m)\, d_c(x_{m-1}, x_m) \tag{5}$$

This implies that

$$d_c(x_n, x_m) \leq p(x_n, x_{n+1})d_c(x_n, x_{n+1})$$

$$+ \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} p(x_j, x_m)\right) p(x_i, x_{i+1})d_c(x_i, x_{i+1})$$

$$+ \prod_{i=n+1}^{m-1} p(x_j, x_m)\, d_c(x_{m-1}, x_m)$$

$$\leq p(x_n, x_{n+1})\lambda^n d_c(x_0, x_1)$$

$$+ \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} p(x_j, x_m)\right) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1)$$

$$+ \prod_{i=n+1}^{m-1} p(x_j, x_m)\, \lambda^{m-1} d_c(x_0, x_1)$$

$$\leq p(x_n, x_{n+1})\lambda^n d_c(x_0, x_1)$$

$$+ \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i} p(x_j, x_m)\right) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1) \tag{6}$$

Let

$$u_r = \sum_{i=0}^{r}\left(\prod_{j=0}^{i} p(x_j, x_m)\right) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1) \tag{7}$$

Consider

$$v_i = \sum_{i=0}^{r}\left(\prod_{j=0}^{i} p(x_j, x_m)\right) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1) \tag{8}$$

In view of condition (2) and the ratio test, we ensure that the series $\sum_i v_i$ converges. Thus, $\lim_{n\to\infty} u_n$ exists. Hence, the real sequence $\{u_n\}$ is Cauchy. Now, using (6), we get

$$d_c(x_n, x_m) \leq d_c(x_0, x_1)[\lambda^n p(x_n, x_{n+1}) + (u_{m-1} - u_n)] \tag{9}$$

Above, we used $p(x,y) \geq 1$. Letting $n,m \longrightarrow \infty$ in (9), we obtain

$$\lim_{n,m\to\infty} d_c(x_n, x_m) = 0 \tag{10}$$

Thus, the sequence $\{x_n\}$ is Cauchy in the complete controlled metric space (X, d_c). So, there is some $x^* \in X$.So that

$$\lim_{n\to\infty} d_c(x_n, x^*) = 0; \tag{11}$$

that is, $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. Now, we will prove that x^* is a fixed point of F. By (1) and condition (iii), we get

$$d_c(x^*, Fx^*) \leq p(x^*, x_{n+1})d_c(x^*, x_{n+1}) + p(x_{n+1}, Fx^*)d_c(x_{n+1}, Fx^*)$$

$$= p(x^*, x_{n+1})d_c(x^*, x_{n+1}) + p(x_{n+1}, Fx^*)d_c(Fx_n, Fx^*)$$

$$\leq p(x^*, x_{n+1})d_c(x^*, x_{n+1})$$

$$+p(x_{n+1}, Fx^*)\left[\gamma_1 \frac{d_c(x^*,Fx^*)[1+d_c(x_n,Fx_n)]}{1+d_c(x_n,x^*)} + \gamma_2 d_c(x_n, x^*)\right]$$

$$= p(x^*, x_{n+1})d_c(x^*, x_{n+1})$$

$$+p(x_{n+1}, Fx^*)\left[\gamma_1 \frac{d_c(x^*,Fx^*)[1+d_c(x_n,x_{n+1})]}{1+d_c(x_n,x^*)} + \gamma_2 d_c(x_n, x^*)\right] \tag{12}$$

Taking the limit as $n \longrightarrow \infty$ and using (10), (11) and the fact that $\lim_{n\to\infty} p(x_n, x)$ and $\lim_{n\to\infty} p(x, x_n)$ exist, are finite, we obtain that

$$d_c(x^*, Fx^*) \leq \left[\gamma_1 \lim_{n\longrightarrow\infty} p(x_{n+1}, Fx^*)\right] d_c(\sigma^*, \mathrm{F}\sigma^*) \tag{13}$$

Suppose that $x^* \neq Fx^*$, having in mind that $\left[\gamma_1 \lim_{n\longrightarrow\infty} p(x_{n+1}, Fx^*)\right] < 1$, so

$$0 < d_c(x^*, Fx^*) \leq \left[\gamma_1 \lim_{n\longrightarrow\infty} p(x_{n+1}, Fx^*)\right] d_c(\sigma^*, \mathrm{F}\sigma^*) < d_c(\sigma^*, \mathrm{F}\sigma^*) \tag{14}$$

It is a contradiction. This yields that $x^* = Fx^*$. Now, we prove the uniqueness of x^*. Let y^* be another fixed point of F in X, then $Fy^* = y^*$. Now, by (1), we have

$$d_c(x^*, y^*) = d_c(Fx^*, Fy^*)$$

$$\leq \gamma_1 \frac{d_c(y^*,Fy^*)[1+d_c(x^*,Fx^*)]}{1+d_c(x^*,y^*)} + \gamma_2 d_c(x^*, y^*)$$

$$= \gamma_1 \frac{d_c(y^*, Fy^*)[1 + d_c(x^*, Fx^*)]}{1 + d_c(x^*, y^*)} + \gamma_2 d_c(x^*, y^*)$$

$$= \gamma_1 \frac{d_c(y^*, y^*)[1 + d_c(x^*, x^*)]}{1 + d_c(x^*, y^*)} + \gamma_2 d_c(x^*, y^*)$$

$$= \gamma_2 d_c(x^*, y^*) \qquad (15)$$

It is a contradiction. This yields that $x^* = y^*$. It completes the proof.

Example

Example 1. Consider $X = \{0,1,2\}$. Take the controlled metric d_c defined as

$$d_c(0,1) = \frac{1}{2}, d_c(0,2) = \frac{11}{20}, d_c(1,2) = \frac{3}{2},$$

where $p: X \times X \longrightarrow [0, \infty)$ is symmetric such that

$$p(0,0) = p(1,1) = p(2,2) = p(1,2) = 1, p(0,2) = 2, p(0,1) = \frac{3}{2}$$

Given $F : X \longrightarrow X$ as

$$F0 = 2 \text{ and } F1 = F2 = 1.$$

Consider$\gamma_1 = \frac{2}{11}, \gamma_2 = \frac{1}{11}$. Then

$$\lambda = \frac{\gamma_2}{1 - \gamma_1} = \frac{\frac{2}{11}}{1 - \frac{1}{11}} = \frac{1}{5} < 1,$$

Take $x_0 = 0$, then $x_1 = 2$, and $x_n = 1$, for all $n \geq 2$, we have

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{p(x_{i+1}, x_{i+2}) p(x_{i+1}, x_m)}{p(x_i, x_{i+1})} = 1 < 5 = \lambda^{-1}$$

Clearly, (2) is satisfied. On the other hand, note that (1) holds for all $x, y \in X$. All other hypotheses of Theorem 1 are verified, and so F has a unique fixed point, which is $x^* = 1$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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