# Fixed Point Results for Rational Type Contraction in $S$-Metric Spaces

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**ABSTRACT**

The goal of this paper is to define rational contraction in the context of $S$-metric spaces and develop various fixed-point theorems in order to elaborate, generalize, and synthesize a number of previously published results. Finally, to illustrate the new theorem, an example is given.

KEYWORDS:$S$-metric space; rational contraction; fixed point.

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## 1. Introduction

Fixed point theory is crucial in science and mathematics. This topic has drawn a lot of interest from academics in the last two decades due to its wide range of applications in disciplines such as nonlinear analysis, topology, and engineering difficulties. The Banach contraction principle [2] is the starting point for most generalizations of metric fixed point theorems. It's difficult to enumerate all of this principle's generalizations. The Banach fixed-point theorem [2] ensures the existence and uniqueness of fixed points of particular self-maps of metric spaces, as well as a constructive approach for discovering them. The S-metric space was introduced by Sedghi et al. [9]. It's a three-dimensional space called the S-metric space. The concept of A-metric space was established by Abbas et al. [1], which is a generalization of S-metric space. Jaggi [7], Das and Gupta [3] discovered the fixed-point theorem for rational contractive type conditions in metric space. The goal of this paper is to define rational contraction in the setting of S-metric spaces, as well as to create various fixed-point theorems to elaborate, generalize, and synthesize several previously published results. Finally, an example is given to demonstrate the new theorem.

## 2. Preliminaries

Some valuable information and ideas will be presented in this section. Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. Sedghi et al. [8, 10] introduced the notion of a $D^*$-metric space as follows.

**Definition 2.1** (see [8, 10]) Let $\mathfrak{D}$ be a non-empty set. A D$^*$-metric on $\mathfrak{D}$ is a function $D^*: \mathfrak{D}^3 \to [0, +\infty)$ that satisfies the following conditions, for each $\xi, \eta, \mu, a \in \mathfrak{D}$;

(D*1). $D^*(\xi, \eta, \mu) \geq 0$,

(D*2). $D^*(\xi, \eta, \mu) = 0$if and only if $\xi = \eta = \mu$.

(D*3). $D^*(\xi, \eta, \mu) = D^*\{\xi, \eta, \mu\}$ (Symmetry in all three variables),

(D*4). $D^*(\xi, \eta, \mu) \leq D^*(\xi, \eta, a) + D^*(a, \mu, \mu)$.

Then $D^*$ is called a $D^*$-metric on $\mathfrak{D}$ and $(\mathfrak{D}, D^*)$ is called a $D^*$-metric space.

**Definition 2.2** (see [9]) Let $\mathfrak{D}$ be a nonempty set. A mapping $S: \mathfrak{D}^3 \to [0,+\infty)$ is called an $S$-metric on $\mathfrak{D}$ if and only if for all $\xi, \eta, \mu, a \in \mathfrak{D}$, the following conditions hold:

(S1). $S(\xi, \eta, \mu) \geq 0$,
(S2). $S(\xi, \eta, \mu) = 0$ if and only if $\xi = \eta = \mu$,
(S3). $S(\xi, \eta, \mu) \leq S(\xi, \xi, a) + S(\eta, \eta, a) + S(\mu, \mu, a)$

The pair $(\mathfrak{D}, S)$ is called an $S$-metric space.

The following is the intuitive geometric example for S-metric spaces.

**Example 2.3** (see [9]) Let $\mathfrak{D} = \mathbb{R}^2$ and d be the ordinary metric on $\mathfrak{D}$. Put

$$S(\xi, \eta, \mu) = d(\xi, \eta) + d(\xi, \mu) + d(\eta, \mu)$$

for all $\xi, \eta, \mu \in \mathfrak{D}$, that is, $S$ is the perimeter of the triangle given by $\xi, \eta, \mu$. Then $S$ is an $S$-metric on $\mathfrak{D}$.

**Example 2.4** Let $\mathfrak{D} = [1,+\infty)$ . Define $S: \mathfrak{D}^3 \to [0,+\infty)$ by

$$S(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^{3} \sum_{i<j} |\eta_i - \eta_j|$$

for all $\eta_i \in X, i = 1,2,3$.

**Lemma 2.5 (see** [9]) Let $(\mathfrak{D}, S)$ be an S-metric space. Then for all $\xi, \eta \in \mathfrak{D}$,

$$S(\xi, \xi, \eta) = S(\eta, \eta, \xi).$$

**Lemma 2.6** Let $(\mathfrak{D}, S)$ be an S-metric space. Then for all $\xi, \eta, \mu \in \mathfrak{D}$,

$$S(\xi, \xi, \mu) \leq 2S(\xi, \xi, \eta) + S(\eta, \eta, \mu) \text{ and}$$

$$S(\xi, \xi, \mu) \leq 2S(\xi, \xi, \eta) + S(\mu, \mu, \eta).$$

**Definition 2.7** (see [9]) Let $\mathfrak{D}$ be an S-metric space.

(i). A sequence $\{\eta_n\}$ converges to $\eta$ if and only if $\lim_{n\to\infty} S(\eta_n, \eta_n, \eta) = 0$. That is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, S(\eta_n, \eta_n, \eta) < \epsilon$ and we denote this by

$$\lim_{n\to\infty} \eta_n = \eta.$$

(ii). A sequence $\{\eta_n\}$ is called a Cauchy if $\lim_{n,m\to\infty} S(\eta_n, \eta_n, \eta_m) = 0$. That is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0, S(\eta_n, \eta_n, \eta_m) < \epsilon$.

(iii). $\mathfrak{D}$ is called complete if every Cauchy sequence in $\mathfrak{D}$ is a convergent.

From (see [9]), we have the following.

**Example 2.8**

(a). Let $\mathbb{R}$ be the real line. Then

$$S(\xi, \eta, \mu) = |\xi - \mu| + |\eta - \mu|$$

for all $\xi, \eta, \mu \in \mathbb{R}$, is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete.

(b). Let $\mathfrak{D}$ be a non-empty set of $\mathbb{R}$. Then

$$S(\xi, \eta, \mu) = |\xi - \mu| + |\eta - \mu|$$

for all $\xi, \eta, \mu \in \mathfrak{D}$, is an S-metric on $\mathfrak{D}$. If $\mathfrak{D}$ is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space $\mathfrak{D}$ is complete.

**Lemma 2.9** (see [9]) Let $(\mathfrak{D}, S)$ be an S-metric space. If the sequence $\{\eta_n\}$ in $\mathfrak{D}$ converges to $\eta$, then $\eta$ is unique.

**Lemma 2.10** (see [9]) Let $(\mathfrak{D}, S)$ be an S-metric space. If

$$\lim_{n\to+\infty} \eta_n = \eta \text{ and } \lim_{n\to+\infty} \mu_n = \mu.$$

Then

$$\lim_{n\to+\infty} S(\eta_n, \eta_n, \mu_n) = S(\eta, \eta, \mu).$$

**Remark 2.11** It is easy to see that every $D^*$-metric is S-metric, but in general the converse is not true, see the following example.

**Example 2.12** Let $\mathfrak{D} = \mathbb{R}^n$ and $\|.\|$ a norm on $\mathfrak{D}$, then

$$S(\xi, \eta, \mu) = \|\eta + \mu - 2\xi\| + \|\eta - \mu\|$$

is S-metric on $\mathfrak{D}$, but it is not $D^*$-metric because it is not symmetric.

The following lemma shows that every metric space is an S-metric space.

**Lemma 2.13** Let $(\mathfrak{D}, d)$ be a metric space. Then we have

(1). $S_d(\xi, \eta, \mu) = d(\xi, \mu) + d(\eta, \mu)$ for all $\xi, \eta, \mu \in \mathfrak{D}$ is an S-metric on $\mathfrak{D}$.

(2). $\lim_{n\to+\infty} \xi_n = \xi$ in $(X, d)$ if and only if $\lim_{n\to+\infty} \xi_n = \xi$ in $(\mathfrak{D}, S_d)$.

(3). $\{\xi_n\}_{n=1}^{\infty}$ is Cauchy in $(\mathfrak{D}, d)$ if and only if $\{\xi_n\}_{n=1}^{\infty}$ is Cauchy in $(\mathfrak{D}, S_d)$.

(4). $(\mathfrak{D}, d)$ is complete if and only if $(\mathfrak{D}, S_d)$ is complete.

**Example 2.14** Let $\mathfrak{D} = \mathbb{R}$ and let

$$S(\xi, \eta, \mu) = |\eta + \mu - 2\xi| + |\eta - \mu|$$

for all $\xi, \eta, \mu \in \mathfrak{D}$. By ([9]), $(\mathfrak{D}, S)$ is an S-metric space. Dung et al. [4] proved that there does not exist any metric d such that

$$S(\xi, \eta, \mu) = d(\xi, \mu) + d(\eta, \mu)$$

for all $\xi, \eta, \mu \in \mathfrak{D}$. Indeed, suppose to the contrary that there exists a metric d with

$$S(\xi, \eta, \mu) = d(\xi, \mu) + d(\eta, \mu)$$

for all $\xi, \eta, \mu \in \mathfrak{D}$. Then

$$d(\xi, \mu) = \tfrac{1}{2} S(\xi, \xi, \mu) = 2|\xi - \mu| \text{ and}$$

$$d(\xi, \eta) = \frac{1}{2} S(\xi, \eta, \eta) = 2|\xi - \eta|$$

for all $\xi, \eta, \mu \in \mathfrak{D}$. It is a contradiction.

In 2012, Sedghi et al. [9] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [9, Remarks 1.3] and [9, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [4] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct.

**Definition 2.15** (see [11]) Let $(\mathfrak{D}, \preccurlyeq)$ be a partially ordered set and let $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be a mapping. Then

1. elements $\eta, \mu \in \mathfrak{D}$ are comparable, if $\eta \preceq \mu$ or $\mu \preceq \eta$ holds;
2. a non empty set $\mathfrak{D}$ is called well ordered set, if every two elements of it are comparable;

3. $\Gamma$ is said to be monotone non-decreasing w.r.t. $\preceq$, if for all $\eta, \mu \in \mathfrak{D}$, $\eta \preceq \mu$ implies $\Gamma\eta \preceq \Gamma\mu$;
4. $\Gamma$ is said to be monotone non-increasing w.r.t. $\preceq$, if for all $\eta, \mu \in \mathfrak{D}$, $\eta \preceq \mu$ implies $\Gamma\eta \succeq \Gamma\mu$.

## 3. Main Results

First, we introduce following definitions.

**Definition 3.1** The triple $(\mathfrak{D}, S, \preccurlyeq)$ is called partially ordered $S$-metric spaces if $(\mathfrak{D}, \preccurlyeq)$ could be a partial ordered set and $(\mathfrak{D}, S)$ be a $S$-metric space.

**Definition 3.2** If $\mathfrak{D}$ is complete $S$-metric, then $(\mathfrak{D}, S, \preccurlyeq)$ is called complete partially ordered metric space.

**Definition 3.3** A partially ordered $S$-metric space $(\mathfrak{D}, S, \preccurlyeq)$ is called an ordered complete (OC), if for each convergent sequence $\{\eta_k\} \subset \mathfrak{D}$, the subsequent condition holds: either

- if $\{\eta_k\} \subset \mathfrak{D}$ is a non-increasing sequence such that $\eta_k \to \eta \in \mathfrak{D}$, then $\eta_k \preccurlyeq \eta$, for all $k \in \mathbb{N}$, that is, $\eta = \inf\{\eta_k\}$, or
- if $\{\eta_k\} \subset \mathfrak{D}$ is a non-decreasing sequence such that $\eta_k \to \eta$ implies that $\eta_k \preccurlyeq \eta$, for all $k \in \mathbb{N}$, that is, $\eta = \sup\{\eta_k\}$.

The following is our first main outcome.

**Theorem 3.1** Let $(\mathfrak{D}, S, \preccurlyeq)$ be a complete partially ordred $S$-metric space. Suppose a self map $\Gamma$ on $\mathfrak{D}$ is continuous, non-decreasing and satisfies the contraction condition

$$S(\Gamma\eta, \Gamma\eta, \Gamma\mu) \leq a\frac{S(\eta,\eta,\Gamma\eta)S(\mu,\mu,\Gamma\mu)}{S(\eta,\eta,\mu)} + b[S(\eta,\eta,\Gamma\eta) + S(\mu,\mu,\Gamma\mu)]$$

$$+cS(\eta,\eta,\mu) + L\min\{S(\eta,\eta,\Gamma\mu), S(\mu,\mu,\Gamma\eta)\} \quad (3.1)$$

for any $\eta \neq \mu \in \mathfrak{D}$ with $\eta \preccurlyeq \mu$, where $L \geq 0$, and $a, b, c \in [0,1)$ with $0 \leq a + 2b + c < 1$. If $\eta_0 \preccurlyeq \Gamma\eta_0$ for certain $\eta_0 \in \mathfrak{D}$, then $\Gamma$ has a fixed point.

**Proof** Define a sequence, $\eta_{k+1} = \Gamma\eta_k$ for $\eta_0 \in \mathfrak{D}$. If $\eta_{k_0+1} = \eta_{k_0}$ for certain $\eta_0 \in \mathbb{N}$, then $\eta_{k_0}$ is a fixed point $\Gamma$. Assume that $\eta_{k+1} \neq \eta_k$ for each $k$. But $\eta_0 \preccurlyeq \Gamma\eta_0$ and $\Gamma$ is non-decreasing as by induction we obtain that

$$\eta_0 \preccurlyeq \eta_1 \preccurlyeq \eta_2 \preccurlyeq \cdots \preccurlyeq \eta_k \preccurlyeq \eta_{k+1} \leq \cdots \quad (3.2)$$

By (3.1), we have

$$\begin{aligned} S(\eta_{k+1}, \eta_{k+1}, \eta_k) &= S(\Gamma\eta_k, \Gamma\eta_k, \Gamma\eta_{k-1}) \\ &\leq a\frac{S(\eta_k,\eta_k,\Gamma\eta_k)S(\eta_{k-1},\eta_{k-1},\Gamma\eta_{k-1})}{S(\eta_k,\eta_k,\eta_{k-1})} \\ &+b[S(\eta_k,\eta_k,\Gamma\eta_k) + S(\eta_{k-1},\eta_{k-1},\Gamma\eta_{k-1})] + cS(\eta_k,\eta_k,\eta_{k-1}) \\ &+L\min\{S(\eta_k,\eta_k,\Gamma\eta_{k-1}), S(\eta_{k-1},\eta_{k-1},\Gamma\eta_k)\} \\ &= a\frac{S(\eta_k,\eta_k,\eta_{k+1})S(\eta_{k-1},\eta_{k-1},\eta_k)}{S(\eta_k,\eta_k,\eta_{k-1})} \\ &+b[S(\eta_k,\eta_k,\eta_{k+1}) + S(\eta_{k-1},\eta_{k-1},\eta_k)] + cS(\eta_k,\eta_k,\eta_{k-1}) \\ &+L\min\{S(\eta_k,\eta_k,\eta_k), S(\eta_{k-1},\eta_{k-1},\eta_{k+1})\} \\ &= a\frac{S(\eta_{k+1},\eta_{k+1},\eta_k)S(\eta_k,\eta_k,\eta_{k-1})}{S(\eta_k,\eta_k,\eta_{k-1})} \\ &+b[S(\eta_{k+1},\eta_{k+1},\eta_k) + S(\eta_k,\eta_k,\eta_{k-1})] + cS(\eta_k,\eta_k,\eta_{k-1}) \\ &= (a+b)S(\eta_{k+1},\eta_{k+1},\eta_k) + (b+c)S(\eta_k,\eta_k,\eta_{k-1}) \end{aligned}$$

which infer that

$$\begin{aligned} S(\eta_{k+1},\eta_{k+1},\eta_k) &\leq \left(\frac{b+c}{1-a-b}\right)S(\eta_k,\eta_k,\eta_{k-1}) \\ &\leq \left(\frac{b+c}{1-a-b}\right)^k S(\eta_1,\eta_1,\eta_0) \leq \cdots \end{aligned} \quad (3.3)$$

For $m, k \in \mathbb{N}$ with $m > k$, by repeated use of (S3), we have

$$\begin{aligned} S(\eta_k,\eta_k,\eta_m) &\leq 2S(\eta_k,\eta_k,\eta_{k+1}) + S(\eta_m,\eta_m,\eta_{k+1}) \\ &\leq 2S(\eta_k,\eta_k,\eta_{k+1}) + S(\eta_{k+1},\eta_{k+1},\eta_m) \end{aligned}$$

$$\leq 2S(\eta_k,\eta_k,\eta_{k+1}) + 2S(\eta_{k+1},\eta_{k+1},\eta_{k+2}) + S(\eta_m,\eta_m,\eta_{k+2})$$

$$\leq 2S(\eta_k,\eta_k,\eta_{k+1}) + 2S(\eta_{k+1},\eta_{k+1},\eta_{k+2}) + S(\eta_{k+2},\eta_{k+2},S\eta_m)$$

$$\leq 2S(\eta_k,\eta_k,\eta_{k+1}) + 2S(\eta_{k+1},\eta_{k+1},\eta_{k+2}) + 2S(\eta_{k+2},\eta_{k+2},\eta_{k+3})$$

$$+S(\eta_m,\eta_m,\eta_{k+3})$$

$$\leq 2S(\eta_k,\eta_k,\eta_{k+1}) + 2S(\eta_{k+1},\eta_{k+1},\eta_{k+2}) + 2S(\eta_{k+2},\eta_{k+2},\eta_{k+3})$$

$$+2S(\eta_{k+3},\eta_{k+3},\eta_{k+4}) + \cdots + 2S(\eta_{m-2},\eta_{m-2},\eta_{m-1})$$

$$+S(\eta_{m-1},\eta_{m-1},\eta_m)$$

$$\leq 2[\lambda^k + \lambda^{k+1} + \cdots + \lambda^{m-2}]S(\eta_0,\eta_0,\eta_1) + \lambda^{m-1}S(\eta_0,\eta_0,\eta_1)$$

$$= 2\lambda^k[1 + \lambda + \lambda^2 + \cdots + \lambda^{m-k-2}]S(\eta_0,\eta_0,\eta_1) + \lambda^{m-k-1}S(\eta_0,\eta_0,\eta_1)$$

$$\leq 2\lambda^k[1 + \lambda + \lambda^2 + \lambda^3 + \cdots]S(\eta_0,\eta_0,\eta_1)$$

$$\leq 2\frac{\lambda^k}{1-\lambda}S(\eta_0,\eta_0,\eta_1) \tag{3.4}$$

where $\lambda = \frac{b+c}{1—a-b}$. As $k,m \to \infty$ in inequality (3.6), we obtain $\lim_{k,m\to\infty} S(\eta_k,\eta_k,\eta_m) = 0$. This shows that $\{\eta_k\} \subset \mathfrak{D}$ is a Cauchy sequence and then $\eta_k \to \zeta \in \mathfrak{D}$ by its completeness. Besides, the continuity of $\Gamma$ implies that

$$\Gamma\zeta = \Gamma\left(\lim_{k\to\infty}\eta_k\right) = \lim_{k\to\infty}\Gamma\eta_k = \lim_{k\to\infty}\eta_{k+1} = \zeta$$

Therefore, $\zeta$ is a fixed point of $\Gamma$ in $\mathfrak{D}$ .

Extracting the continuity of a map $\Gamma$ in Theorem 3.1, we have the below result.

**Theorem 3.2** If $\mathfrak{D}$ has an ordered complete (OC) property in Theorem 3.1, then a non-decreasing mapping $\Gamma$ has a fixed point in $\mathfrak{D}$.

***Proof*** We only claim that $\Gamma\zeta = \zeta$. By an ordered complete metrical property of $\mathfrak{D}$, we have $\zeta = \sup\{\eta_k\}$, for $k \in \mathbb{N}$ as $\eta_k \to \zeta \in \mathfrak{D}$ is a non-decreasing sequence. The non-decreasing property of a map $\Gamma$ implies that $\Gamma\eta_k \preccurlyeq \Gamma\zeta$ or, equivalently, $\eta_{k+1} \preccurlyeq \Gamma\zeta$, for $k \geq 0$. Since, $\eta_0 \prec \eta_1 \preccurlyeq \Gamma\zeta$ and $\zeta = \sup\{\eta_k\}$ as a result, we get $\zeta \preccurlyeq \Gamma\zeta$. Assume $\zeta \prec \Gamma\zeta$. From Theorem 3.1, there is a non-decreasing sequence $\Gamma^k\zeta \in \mathfrak{D}$ with $\lim_{k\to\infty} \Gamma^k\zeta = \varepsilon \in \mathfrak{D}$. Again, by an ordered complete (OC) property of $\mathfrak{D}$, we obtain that $\varepsilon = \sup\{\Gamma^k\zeta\}$. Furthermore, $\eta_k = \Gamma^k\eta_0 \preccurlyeq \Gamma^k\zeta$, for $k \geq 1$ as a result, $\eta_k \prec \Gamma^k\zeta$, for $k \geq 1$, since $\eta_k \preccurlyeq \zeta \prec \Gamma\zeta \preccurlyeq \Gamma^k\zeta$, for $k \geq 1$ whereas $\eta_k$ and $\Gamma^k\zeta$, for $k \geq 1$ are distinct and comparable.

Now we have the discussion below in the subsequent cases.

Case-1 If $S(\eta_k, \eta_k, \Gamma^k\zeta) \neq 0$, then (3.1) becomes,

$$\begin{aligned}
S(\eta_{k+1}, \eta_{k+1}, \Gamma^{k+1}\zeta) &= S\left(\Gamma\eta_k, \Gamma\eta_k, \Gamma(\Gamma^k\zeta)\right) \\
&\leq a\frac{S(\eta_k,\eta_k,\Gamma\eta_k)S(\Gamma^k\zeta,\Gamma^k\zeta,\Gamma^{k+1}\zeta)}{S(\eta_k,\eta_k,\Gamma^k\zeta)} \\
&+b\left[S(\eta_k,\eta_k,\Gamma\eta_k) + S(\Gamma^k\zeta,\Gamma^k\zeta,\Gamma^{k+1}\zeta)\right] \\
&+cS(\eta_k,\eta_k,\Gamma^k\zeta) \\
&+L\min\{S(\eta_k,\eta_k,\Gamma^{k+1}\zeta), S(\Gamma^k\zeta,\Gamma^k\zeta,\Gamma\eta_k)\} \\
&= a\frac{S(\eta_k,\eta_k,\eta_{k+1})S(\Gamma^k\zeta,\Gamma^k\zeta,\Gamma^{k+1}\zeta)}{S(\eta_k,\eta_k,\Gamma^k\zeta)} \\
&+b\left[S(\eta_k,\eta_k,\eta_{k+1}) + S(\Gamma^k\zeta,\Gamma^k\zeta,\Gamma^{k+1}\zeta)\right] \\
&+cS(\eta_k,\eta_k,\Gamma^k\zeta) \\
&+L\min\{S(\eta_k,\eta_k,\Gamma^{k+1}\zeta), S(\Gamma^k\zeta,\Gamma^k\zeta,\eta_{k+1})\} \qquad (3.5)
\end{aligned}$$

As $k \to \infty$ in (3.5), we get

$$\begin{aligned}
S(\zeta,\zeta,\varepsilon) &\leq cS(\zeta,\zeta,\varepsilon) + L\min\{S(\zeta,\zeta,\varepsilon), S(\varepsilon,\varepsilon,\zeta)\} \\
&\leq (c+L)S(\zeta,\zeta,\varepsilon)
\end{aligned}$$

as a result, we have, $S(\zeta,\zeta,\varepsilon)=0$. Hence $\zeta=\varepsilon$. In particular, $\zeta=\varepsilon=\sup\{\Gamma^{k}\zeta\}$ in consequence, we get $\Gamma\zeta\preccurlyeq\zeta$, a contradiction. Therefore,$\Gamma\zeta=\zeta$.

Case-2 Case-1 If $S(\eta_k,\eta_k,\Gamma^{k}\zeta)=0$, then, $S(\zeta,\zeta,\varepsilon)=0$ as $k\to\infty$. By following the similar argument in Case 1, we get $\Gamma\zeta=\zeta$.

**Corollary 3.1** Let $(\mathfrak{D},S,\preccurlyeq)$ be a complete partially ordred $S$-metric space. Suppose a self map $\Gamma$ on $\mathfrak{D}$ is continuous, non-decreasing and satisfies the contraction condition

$$S(\Gamma\eta,\Gamma\eta,\Gamma\mu)\leq a\frac{S(\eta,\eta,\Gamma\eta)S(\mu,\mu,\Gamma\mu)}{S(\eta,\eta,\mu)}+b[S(\eta,\eta,\Gamma\eta)+S(\mu,\mu,\Gamma\mu)]+cS(\eta,\eta,\mu) \tag{3.6}$$

for any $\eta\neq\mu\in\mathfrak{D}$ with $\eta\preccurlyeq\mu$, where $a,b,c\in[0,1)$ with $0\leq a+2b+c<1$. If $\eta_0\preccurlyeq\Gamma\eta_0$ for certain $\eta_0\in\mathfrak{D}$, then $\Gamma$ has a fixed point.

*Proof.* It follows by $L=0$ in Theorem 3.1.

**Corollary 3.2** Let $(\mathfrak{D},S,\preccurlyeq)$ be a complete partially ordred $S$-metric space. Suppose a self map $\Gamma$ on $\mathfrak{D}$ is continuous, non-decreasing and satisfies the contraction condition

$$S(\Gamma\eta,\Gamma\eta,\Gamma\mu)\leq a\frac{S(\eta,\eta,\Gamma\eta)S(\mu,\mu,\Gamma\mu)}{S(\eta,\eta,\mu)}+cS(\eta,\eta,\mu) \tag{3.7}$$

for any $\eta\neq\mu\in\mathfrak{D}$ with $\eta\preccurlyeq\mu$, where $L\geq0$, and $a,b,c\in[0,1)$ with $0\leq a+2b+c<1$. If $\eta_0\preccurlyeq\Gamma\eta_0$ for certain $\eta_0\in\mathfrak{D}$, then $\Gamma$ has a fixed point.

*Proof.* Taking $b=0,L=0$ in Theorem 3.1, we obtain the desired result.

We conclude with an example.

**Example 3.1** Let $(\mathbb{R},S,\preccurlyeq)$ be a totally ordered complete $S$-metric space with $S$-metric defined as in Example 2.8 (a). Let $\Gamma:\mathbb{R}\to\mathbb{R}$ be a map defined by $\Gamma(\eta)=\frac{3\eta+24\mathrm{n}-3}{24\mathrm{n}}$ for all $n\geq 1$. It is evident that $\Gamma$ is continuous and non-decreasing in $\mathbb{R}$ and $\eta_0=0\in\mathbb{R}$ such that $\eta_0=0\preccurlyeq\Gamma\eta_0$. Taking $a=0,b=0,c=\frac{1}{n},L=0$. For $\eta\preccurlyeq\mu$, we have

$$S(\Gamma\eta,\Gamma\eta,\Gamma\mu)=2|\Gamma\eta-\Gamma\mu|$$

$$=2\left|\frac{3\eta+24\mathrm{n}-3}{24\mathrm{n}}-\frac{3\mu+24\mathrm{n}-3}{24\mathrm{n}}\right|$$

$$= 2\left|\frac{3(\eta-\mu)}{24\text{n}}\right| = \frac{1}{n}\left|\frac{\eta-\mu}{4}\right|$$

$$\leq \frac{1}{n}|\eta-\mu| = \frac{1}{n}S(\eta,\eta,\mu)$$

$$\leq a\frac{S(\eta,\eta,\Gamma\eta)S(\mu,\mu,\Gamma\mu)}{S(\eta,\eta,\mu)} + b[S(\eta,\eta,\Gamma\eta) + S(\mu,\mu,\Gamma\mu)]$$

$$+cS(\eta,\eta,\mu) + L\min\{S(\eta,\eta,\Gamma\mu), S(\mu,\mu,\Gamma\eta)\}$$

holds for every $\eta,\mu \in \mathbb{R}$. For $L \geq 0$ and $a,b,c \in [0,1)$ such that $0 \leq a+2b+c < 1$, in particular, if we take $a=0, b=0, c=\frac{1}{n}, L=0$, then $0 \leq a+2b+c < 1$ and $1 \in \mathbb{R}$ is a fixed point of $\Gamma$ as all the conditions of Theorem 3.1 are satisfied.

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